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CONTENTS

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MINIMAL COVERS FOR CLOSED CURVES

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I. Introduction. In a recent note **[1]**, Nitsche had given a short proof of the following:

THEOREM 1. *Any continuous closed space curve C of length L is contained in a ball of radius $R \leqq L/4$, and equality is forced only if C is a "needle", i.e., a line segment of length $L/2$ traversed twice.*

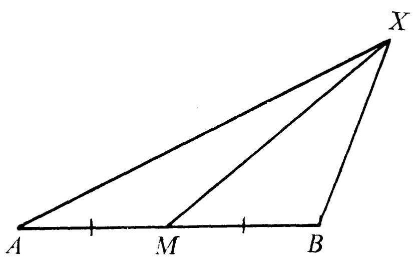
Thus any closed curve of length 4 can be "covered" by a ball of radius 1 and no smaller ball will suffice in general. In this paper, we shall consider some related covering problems. For example, we shall show that any closed curve of length 4 can be covered by a cube of diagonal 2. Since such a cube has a circumscribed sphere of radius 1, this gives another proof of Theorem 1. We shall also consider the problem of covering a closed curve with a set of given type having minimum measure. For example, what is the minimum area of a rectangle containing a copy of every plane closed curve of length 4?

It turns out that Theorem 1 was first proved in a more general form by Segre [**2**] and independently by Rutishauser and Samelson [**3**]. In Section 2, we shall give a proof of Theorem 1 which is the same as the elegant proofs in [**2**] and [**3**] but set in the context of Euclidean space and discuss briefly the relationship with Fenchel's theorem on space curves. In Section 3, we shall find the minimal cube containing a closed curve of length 4. The last section (4) deals with special covering problems for plane curves.

2. Proof of Theorem 1. Let $A \neq B$ be two points on C dividing it into two arcs of equal length $L/2$, and let M be the midpoint of the segment from A to B. For $X \in C$, we have

$$MX \leqq \tfrac{1}{2}(AX + BX) \leqq L/4. \tag{1}$$

To see why the first inequality in (1) is true, consider the following figure.



By central reflection of X through M to a collinear point X^*, with $MX^* = MX$, we see that (1) is a consequence of the triangle inequality applied to $\triangle XAX^*$. It follows from (1) that C is contained in the ball of radius $L/4$ centered at M. Moreover, one readily deduces that equality can hold in (1) only if X is collinear with A and B. Thus if C is contained in no smaller ball, then it must pass through the endpoints of a diameter, in which case C can only be a needle. This completes the proof.

In [**2**] and [**3**], the theorem is actually proved in the following form.

THEOREM 2. *Let C be a continuous closed curve of length $L < 2\pi$ on the unit sphere $\{X: |X| = 1\}$ in Euclidean n-space. Then C is contained in a spherical disk of spherical radius $R \leqq L/4$, and equality is forced only if C is an arc of a great circle traversed twice.*

As an immediate consequence, one has

COROLLARY 1. *Any closed curve of length less than 2π on the unit sphere is contained in an open hemisphere.*

This corollary leads to a proof of Fenchel's theorem that the total curvature of any closed space curve is at least 2π. An elementary exposition of this, together with a neat proof of Corollary 1 using an idea similar to that used for proving Theorem 1, is given by Horn [**4**]. Fenchel's article [**5**] is an excellent survey of these matters. One finds another excellent exposition, with the proof of Theorem 2 in the case $n = 3$, in an article of Chern [**6**]. Still another proof of Theorem 2 is given by Besicovitch [**7**].

The proof of Theorem 2 given in [**2, 3**] is the same as that of Theorem 1, with Euclidean distances replaced by spherical distances. (In this case one knows that (1) is valid whenever $MX < \pi/2$. Thus if $MX < \pi/2$, then in fact $MX \leqq L/4 < \pi/2$. Hence MX, as a continuous function of X on the connected set C, cannot take values in the open interval $(L/4, \pi/2)$. But $MX < \pi/2$ when $X = A$, so all values of MX are $\leqq L/4$.) The proof generalizes to a large class of metric spaces, including Euclidean space, hyperbolic space, and indeed any symmetric space in the sense of Cartan.

3. The smallest cube covering a closed curve. In this section, we prove that a closed curve of length 4 is contained in a cube of major diagonal length 2. This will follow from the next theorem and the fact that every closed and bounded set admits a circumscribed cube.

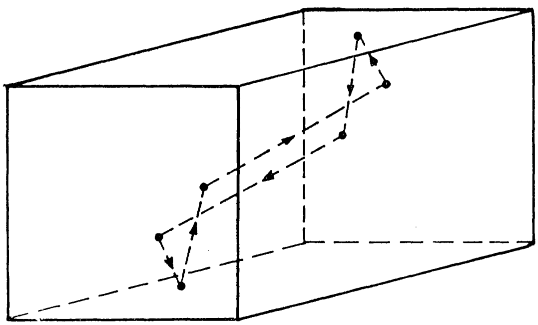
THEOREM 3. *Let C be a closed curve of length 4 and B any rectangular box circumscribed about C (so each face of B touches C). Then the major diagonal of B has length $D \leqq 2$.*

Proof. Introduce rectangular (x, y, z)-coordinates with the origin at one corner of B and axes along edges of B. Suppose B has edges of lengths a, b, and c. Then, if s represents arc length along C, we have

$$(2)\qquad 2D^2 = 2(a^2+b^2+c^2) \leqq \oint_C \left\{ a \left| \frac{dx}{ds} \right| + b \left| \frac{dy}{ds} \right| + c \left| \frac{dz}{ds} \right| \right\} ds$$

$$\leqq \oint_C \{a^2+b^2+c^2\}^{\frac{1}{2}} \left\{ \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 \right\}^{\frac{1}{2}} ds = 4D,$$

so that $D \leqq 2$. The first inequality in (2) follows since C touches each face of B, and the second follows from the Cauchy-Schwarz inequality. Equality can hold throughout (2) only if the tangent vector $(dx/ds, dy/ds, dz/ds)$ of C is parallel to $(\pm a, \pm b, \pm c)$ almost everywhere and C touches each face of B just once. This restricts C to a class of "chairshaped hexagons" inscribed in B with edges parallel to major diagonals of B. These correspond to closed paths of light rays reflecting from the walls of B. The following figure shows an example. These hexagons, in-



cluding the degenerate cases of certain parallelograms and major diagonals traversed twice, all have length $2D$. A more elementary proof can be given similar to the way one finds the shortest four cushion shot on a billiard table; one tiles space with boxes and draws a line parallel to a diagonal. Interesting discussions of these polygons, with related problems about polygons reflecting inside polyhedra are given by Jacobson and Yocom [**8**], Jacobson [**9**] and Gardner [**10**].

Observe that Theorem 1 follows from Theorem 3, since a box of major diagonal $L/2$ has circumscribed sphere of radius $L/4$. Note also that these theorems, and their proofs, extend to Euclidean n-space for any $n \geqq 2$.

A theorem of Kakutani [**11**] implies that any closed and bounded subset S of 3-space admits a circumscribed cube, that is, a cube containing S with each face meeting S. By Theorem 3, a cube circumscribed about a closed curve of length 4 has diagonal length at most 2. Hence we have the following.

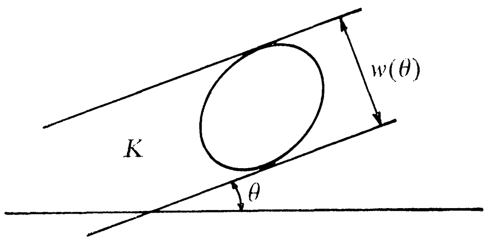
THEOREM 4. *Any closed space curve of length* 4 *is contained in a cube of major diagonal length* 2.

The extension of Kakutani's theorem to any dimension is valid (see [**12**]), hence Theorem 4 holds in Euclidean n-space for any $n \geqq 2$.

4. Minimal covers for plane curves. Here, we consider the problem of finding a plane convex set of minimum area that will cover any plane closed curve of length 4. By the plane result corresponding to Theorem 1, any such curve can be covered by a circular disk of area π. On the other hand, the plane case of Theorem 4 shows that a square of area 2 will cover any closed plane curve of length 4. The following theorem yields the rectangle of smallest area that suffices to cover any closed plane curve of length 4.

THEOREM 5. *The rectangle of minimum area (having a fixed shape) that suffices to cover every plane closed curve of length* 4 *is that rectangle having diagonal length* 2 *and a side of length* $4/\pi$.

Proof. Any rectangle that will serve to cover all plane closed curves of length 4 must have diagonal length at least 2 (to accommodate a line segment of length 2) and sides longer than $4/\pi$ (to accommodate a circle of perimeter 4). It is easy to see that of all such rectangles, the rectangle of diagonal 2 and one side equal to $4/\pi$ has the least area. We must now show that this rectangle will actually cover all plane closed curves of length 4. To do this, it is convenient to use the following result due to Cauchy. Let K be plane convex curve of length L. For each angle θ, let $w(\theta)$ be the width of K in the direction θ. That is, $w(\theta)$ is the distance between the two parallel supporting lines of K making angle θ with the horizontal.



Then the formula of Cauchy is,

$$L = \tfrac{1}{2}\int_0^{2\pi} w(\theta)d\theta = \pi\bar{w}, \tag{3}$$

where $\bar{w}$ denotes the average value of $w(\theta)$ on $[0, 2\pi]$. Now observe that it suffices to prove our result for convex curves. For if C is a closed plane curve of length 4, then the smallest convex curve K containing C has length $\leqq 4$. If K can be covered by a rectangle of the required type, then so can C. It is also clear that it suffices to consider convex curves of length exactly 4. If K is such a curve, then by (3) the average width of K is $4/\pi$; hence the width in some direction is $4/\pi$. The rectangle R circumscribed about K and having two sides parallel to this direction has a side of length $4/\pi$ and, by the plane case of Theorem 3, diagonals of length $\leqq 2$. It is easy to see that R is contained in a rectangle of side $4/\pi$ and diagonals of length 2. This completes the proof.

The next theorem, proved by Schaer [**13**], shows that if we allow rectangles of variable shape, then there is always a rectangle of area less than that of the rectangle in Theorem 5 covering a given closed plane curve of length 4.

THEOREM 6. *Any plane closed curve of length* 4 *is contained in some rectangle of area* $16/\pi^2$.

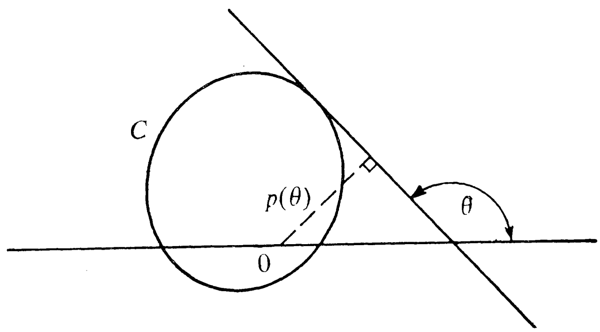
Proof. As in the proof of Theorem 5, one sees that it suffices to prove this theorem for convex curves of length 4. Applying Cauchy's formula (3) to a convex curve K of length 4 and using the A.M.–G.M. inequality, we have,

$$(4)\qquad 4 = \tfrac{1}{2}\int_0^{2\pi} w(\theta)d\theta = \tfrac{1}{4}\int_0^{2\pi} \{w(\theta) + w(\theta+\pi/2)\}d\theta$$
$$\geqq \tfrac{1}{2}\int_0^{2\pi} \sqrt{w(\theta)w(\theta+\pi/2)}\,d\theta \geqq \pi \min \sqrt{w(\theta)w(\theta+\pi/2)}.$$

Hence $w(\theta)w(\theta+\pi/2) \leqq 16/\pi^2$ for some θ, which yields a rectangle of area $\leqq 16/\pi^2$ circumscribed about K. The theorem follows.

Note that Theorem 6 is sharp, since a circle of perimeter 4 does not admit a circumscribed rectangle of area less than $16/\pi^2$.

Finally, we consider triangular covers. As proved by Wetzel [**14**], the triangle of minimum area (having a fixed shape) that suffices to cover every plane closed curve of length 4 is the equilateral triangle of side $4\sqrt{3}/\pi$. (This has area greater than the minimal covering rectangle in Theorem 5.) If we restrict our attention to equilateral triangles, then an averaging type of argument, similar to that used in proving Theorem 6, can be used to prove that every plane closed curve of length 4 is contained in an equilateral triangle of side $4\sqrt{3}/\pi$. Indeed, it suffices to consider convex curves of length 4. If C is such a curve and 0 a point inside, let $p(\theta)$ denote the distance from 0 to the supporting line making angle θ with the horizontal, as indicated in the figure below.



Then a stronger form of Cauchy's formula (3) asserts that

$$(5)\qquad \text{length of } C = \int_0^{2\pi} p(\theta)d\theta.$$

Observe that since $w(\theta) = p(\theta) + p(\theta+\pi)$, formula (3) is an immediate consequence of (5). For each θ, the supporting lines corresponding to $\theta, \theta + (2\pi/3)$, $\theta + (4\pi/3)$ form an equilateral triangle circumscribed about C. Since the sum of the lengths of the perpendiculars drawn from 0 to the sides is equal to the altitude of the triangle, we have

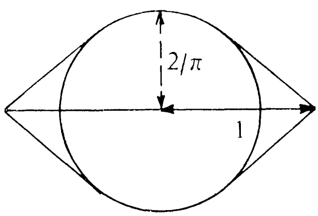
$$p(\theta)+p\left(\theta+\frac{2\pi}{3}\right)+p\left(\theta+\frac{4\pi}{3}\right)=\frac{\sqrt{3}}{2}s(\theta), \tag{6}$$

where $s(\theta)$ is the side of the circumscribed triangle. The integral over $[0,2\pi]$ of the left hand side of (6) equals three times the length of C, that is 12. Hence the average of $s(\theta)$ over $[0,2\pi]$ is $4\sqrt{3}/\pi$. Thus some triangle has side $4\sqrt{3})\pi$, as we wanted to prove.

Using a generalization of the previous averaging type of argument, Eggleston **[15]** has shown more generally that the smallest triangle with given angles α,β,γ that covers all closed curves of given length is the one circumscribed about a circle of that length.

The example of a circle of length 4 shows that no equilateral triangle of side less than $4\sqrt{3}/\pi$ suffices to cover all closed curves of length 4.

This brings us to the question of determining a plane convex set U of minimum area that will suffice to cover any plane closed curve of length 4. This problem is, to our knowledge, unsolved although it is possible to obtain lower bounds on the area of U. For example, U must contain a line segment of length 2 and a circle of perimeter 4, hence it must contain the smallest convex set containing the union of the segment and circle. It is possible to show that the area of the smallest convex set containing copies of two centrally symmetric convex sets is minimized when their centers coincide. Hence a lower bound for the area of U is given by the area of the smallest convex set containing a circle of radius $2/\pi$ and a line segment of length 2 through its center.



5. Open problems. The extension of Theorem 6 to 3-space would be to find the smallest volume rectangular box which can cover any closed curve of length 4. We can show that a box of volume 1 suffices but we do not know whether this bound is sharp. The proof depends on showing that the convex hull of C has average width $\leqq 1$. It can then be shown that the convex hull is contained in some box of edge lengths a,b,c satisfying $a+b+c\leqq 3$, hence, volume $\leqq 1$. One can next ask for the minimal regular tetrahedral cover. We also have another set of related problems by changing the class C of closed curves to open curves. A two-dimensional result of the latter type was set in the 1969 William Lowell Putnam Mathematics Competition **[16]**, i.e., "Show that any curve of unit length can be covered by a closed rectangle of area 1/4." The three-dimensional version was posed in **[17**, p. 74**]**. Other two-dimensional results are given in a paper of Wetzel **[18]** and a report of Schaer **[19]**.

Authors' Note: Since this paper was submitted for publication, it has come to our attention that these results have been discovered independently by J. E. Wetzel (*Covering balls for curves of constant length*; L'Enseignement Math., 17 (1971) 275–277) and J. Schaer and J. E. Wetzel (*Boxes for curves of constant length*; accepted for publication, Israel J. Math.). In the latter paper, there is additionally a proof of the theorem: *Every arc of length L in E^n lies in some semiball of radius $L/2$, but no smaller semiball contains a congruent copy of every arc of length L.*

References

1. J. C. C. Nitsche, The smallest sphere containing a rectifiable curve, Amer. Math. Monthly, 78 (1971) 881–882.

2. B. Segre, Sui circoli geodetici di una superficie a curvatura totale costante, che contengono nell'interno una linea assegnata, Boll. Un. Mat. Ital., 13 (1934) 279–283.

3. H. Rutishauser and H. Samelson, Sur le rayon d'une sphère dont la surface contient une courbe fermée, C. R. Acad. Sci. Paris, 227 (1948) 755–757.

4. R. A. Horn, On Fenchel's theorem, Amer. Math. Monthly, 78 (1971) 380–381.

5. W. Fenchel, On the differential geometry of closed space curves, Bull. Amer. Math. Soc., 57 (1951) 44–54.

6. S. S. Chern, Curves and surfaces in Euclidean space, Studies in Global Geometry and Analysis, MAA, 1967.

7. A. S. Besicovitch, A net to hold a sphere, Math. Gaz., 41 (1957) 106–107.

8. R. A. Jacobson and K. L. Yocom, Paths of minimal length within a cube, Amer. Math. Monthly, 73 (1966) 634–639.

9. R. A. Jacobson, Paths of minimal length within hypercubes, Amer. Math. Monthly, 73 (1966) 868–872.

10. Martin Gardner, Sixth Book of Mathematical Games from Scientific American, Freeman, San Francisco, 1971.

11. S. Kakutani, A proof that there exists a circumscribing cube around any bounded closed convex set in R^3, Ann. of Math., 43 (1942) 739–741.

12. H. Yamabe and Z. Yujobô, On the continuous functions defined on a sphere, Osaka Math. J., 2 (1950) 19–22.

13. J. Schaer, Lecture given at the Conference on Combinatorial Geometry, Michigan State University, East Lansing, 1966.

14. J. E. Wetzel, Triangular covers for closed curves of constant length, Elem. Math., 25 (1970) 78–82.

15. H. G. Eggleston, Problems in Euclidean Space, Pergamon, New York, 1957, p. 157.

16. Solution, Amer. Monthly, 77 (1970) 724–727.

17. M. S. Klamkin, On the ideal role of an industrial mathematician and its educational implications, Amer. Math. Monthly, 78 (1971) 53–76.

18. J. E. Wetzel, On Moser's problem of accommodating closed curves in triangles, Elem. Math., 27 (1972) 35–36.

19. J. Schaer, The broadest curve of length 1, University of Calgary, Department of Mathematics Research Paper No. 52.

SEPARATING POINTS IN A RECTANGLE

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1. Problem statement. In the Journal of Recreational Mathematics of July 1969, problem 88 [9] asks what is the minimum distance between any pair of points in a set of N points which are placed in the closed unit square to maximize this minimum distance. The arrangement (or arrangements) of the N points that achieves this separation is also desired.

Solutions are known for $N = 2, 3, 4, 5, 6, 8$, and 9 [**6, 7, 8,**]. (For $N = 7$, Ref. [7] contains an arrangement and an assertion that the author has privately proved it to be optimal. However no published proof is known to this writer.) For larger values of N, M. Goldberg [**2**] has empirically obtained lower bounds by finding good arrangements for all N up to 30, and for selected higher values of N up to 340, J. Schaer [**5**] improved Goldberg's result for $N = 10$. J. S. Byrnes [**1**] has noted that substantial and unexpected theoretical difficulties arise in considering this problem, making the investigations quite complicated even for moderate values of N. (The author's proof for $N = 6$ requires 10 printed pages [**8**].)

There are many obvious generalizations to this problem. One could consider separating points in a 3-dimensional or n-dimensional cube, in an arbitrary plane polygon, in a circle [**3, 4**], or indeed in any arbitrary region. The present paper considers a very simple extension to the original problem, and even this rapidly leads to some subtle features. Our problem will be to place N points in an arbitrary *rectangle* to maximize the minimum separation between any two. We shall solve the cases $N = 2$, 3, 4, and 5.

Our methods are entirely self-contained; there will be no appeal to the known solutions for the square, although of course the ideas used in the prior solutions have been suggestive to us. The solutions for the square will emerge as special cases from the present investigation, providing a partial check.

We shall normalize the problem by taking as the unit of measure the longer side of the constraining rectangle. Hence the problem can be stated as maximizing the separation of N points in a rectangle, 1 by A units, where $0 < A \leqq 1$.

2. A lemma, and the solution for $N = 2$. The problem is clearly meaningless for $N = 1$. When $N \geqq 2$, we have a simple but surprisingly powerful lemma.

LEMMA. *For any N, there is an optimal arrangement in which there is a point on each of the four sides of the rectangle, if we agree that each vertex is on both sides which meet there.*

Proof. Suppose any arrangement is given in which there is (for example) no point on the right hand edge. Select the right-most point in the given arrangement, and move it horizontally to the right until it bumps into the rectangle edge. This move cannot decrease the separation between any points, and in fact must increase the separation between the point moved and all others. Hence the new arrangement obtained must have at least the same minimum separation, and furthermore has a point on the right hand edge. This proves the lemma. A similar proof was given in **[6]**.

In the remainder of the paper, we shall limit consideration to solutions with points on all edges. Note that all we have proved is that some optimal arrangement meets these conditions, not that all of them must. The latter is in fact true for the cases we consider in this paper, but we neither need that fact nor claim to have proved it.

We are now ready to solve the case $N = 2$. If there are only two points in the arrangement, (that is, when $N=2$) the only way to obtain points on all four edges, as the lemma proves is possible, is to have each of the two do double duty by occupying a corner. Hence we have a rigorous proof that the diagonally opposite corners are the positions occupied by two points in a rectangle with maximum separation. Of course, this was intuitively clear from the beginning.

3. Saw-tooth solution. For larger values of N, there are also some general statements. If A is small, a general technique provides the answer at once. We illustrate with $N = 5$, $A \leqq \sqrt{3}/4$. In Figure 1, we see a rectangle, 1 by .4, in which five points are to be located. The rectangle is subdivided by vertical lines into four congruent rectangular regions, each .25 by .4.

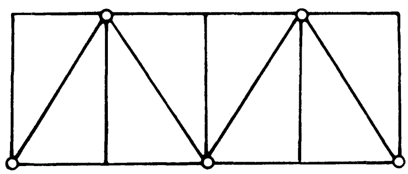

FIG. 1.

Since there are four subregions and five points, we conclude that at least one (closed) subregion must contain two of the points. In that region we can apply the result for $N = 2$ to conclude that the separation cannot exceed $d = \sqrt{(.4^2 + .25^2)} = .47$ which is the diagonal of the subregion. Hence the entire arrangement of five points must have at least one pair with separation no greater than d. So d is an upper bound on the separation attainable with $N = 5$ in the rectangle of Figure 1.

But in Figure 1, the five points are actually located in a zigzag arrangement which achieves this separation (or greater) for all pairs. Hence d is attainable, and Figure 1 is an optimal arrangement. We shall call such an arrangement the *saw-tooth* solution.

Specifically, the saw-tooth solution has points located alternately at the bottom and top of equally spaced vertical dividers in the rectangle. It is easy to show that the saw-tooth solution is optimal for any N whenever $A \leqq \sqrt{3}/(N-1)$. In the following sections, we therefore shall consider values of A satisfying $\sqrt{3}/(N-1) < A \leqq 1$.

4. Three points. Consider now the case $N = 3$. From the lemma, we know that one point must be at a corner, since otherwise, not all four edges would be occupied. Say the corner point is in the lower left, labelled P in Figure 2. Furthermore, the points Q and R must be on the top and right hand edges respectively, also by the lemma.

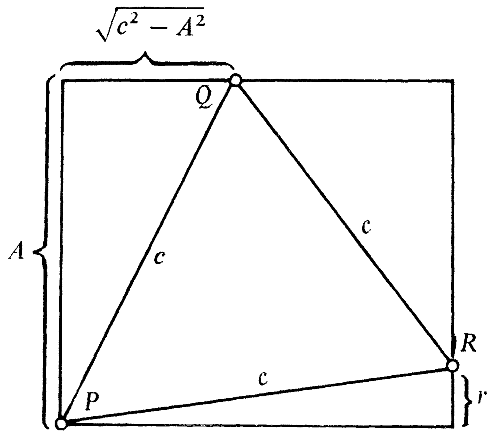

FIG. 2.

If $A \leqq \sqrt{3}/2 = .866$, the saw-tooth solution applies, with R in the lower right corner. If $A > \sqrt{3}/2$, then we must consider positions for R above the corner.

Let R be r units up from the bottom edge. We claim that it now suffices to consider arrangements in which the segments PQ and QR are equal. For if they are not equal, then Q can be moved to right or left until equality is achieved, and the minimum separation will not be reduced by such a move. For example, if originally $PQ < QR$, then Q can be moved to the right to a new position Q' where $PQ < PQ' = Q'R$.

Therefore only arrangements with $PQ = QR$ are of interest. For any value of r, such an arrangement is uniquely determined. Consider the minimum separation d attained, as a function of r, with r increasing from 0 to A. Initially, when $r = 0$, we have a sawtooth arrangement, and PR is the shortest leg of the triangle (since $A > \sqrt{3}/2$). As r increases, PR increases, and the other two (equal) sides decrease, until the three become equal. Then with further increase in r, the sides PQ and QR are shortest, and ultimately attain the value $(1/2)(1 + A^2)$ when R reaches the top right corner, in Figure 3.

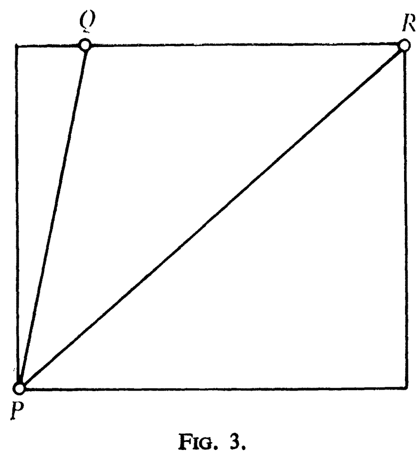

FIG. 3.

It is clear that the minimum separation is maximized when ΔPQR is equilateral. A conceptual procedure for finding the optimal arrangement is the following: With P as center, strike an arc of (varying) radius $c; c \geqq 1$. This arc intersects the top and right side respectively of the rectangle in the points Q and R. Beginning with $c = 1$, increase c until the segment QR has length c. The resulting arrangement (P,Q,R) is optimal.

Referring to Figure 2 one sees that the desired value of c satisfies the algebraic equation $c^2 = (A - \sqrt{c^2 - 1})^2 + (1 - \sqrt{c^2 - A^2})^2$. This is easily reduced to a quartic equation in c^2; or it can be solved directly numerically from the above equation by any of the standard methods when the value of A is specified.

5. Orbits. It is of interest to see how the points P, Q, and R move as a function of A. In Figure 4 the optimal arrangement for a square is shown ($A = 1$). As A decreases toward $\sqrt{3}/2$, point P remains fixed; point Q follows a curve in the interior of the square (always at height A above the bottom) to a point on the vertical bisector of the square; and point R moves down the right hand side to the lower right corner. As A decreases further, P and R remain fixed, and Q descends along the vertical bisector.

These paths are shown schematically in Figure 4. We shall refer to the locus of a point in the solution (with varying A) as the *orbit* of the point. This definition is not precise, and we shall not attempt to make it so, since the idea is only descriptive. The examples will always clarify the intent in context.

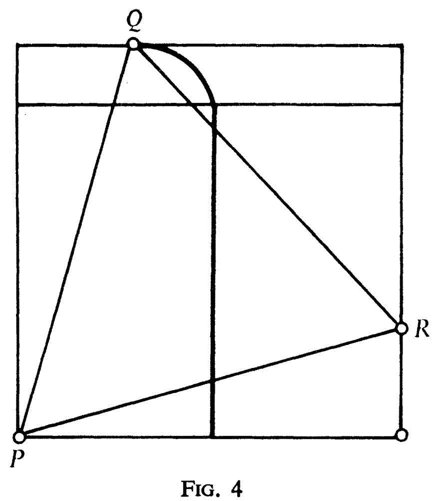

FIG. 4

6. Five points. By a quirk this writer cannot explain, it appears easiest now to skip the case $N = 4$ until later, and turn our attention to the case $N = 5$. We have at once, if $A \leqq \sqrt{3}/4 = .433$, the saw-tooth solution.

When A is nearly 1, we have a solution by a similar simple device. The rectangle is divided into four congruent subrectangles (as in the saw-tooth proof), but this time, by using one vertical and one horizontal bisector. Figure 5 shows the construction with $A = .9$. Arguing as before that there must be two points in at least one subrectangle, we establish the diagonal $d = \sqrt{(.45^2+.5^2)}$ as an upper bound. And in Figure 5, that bound is attained by taking the four corners and the center of the rectangle.

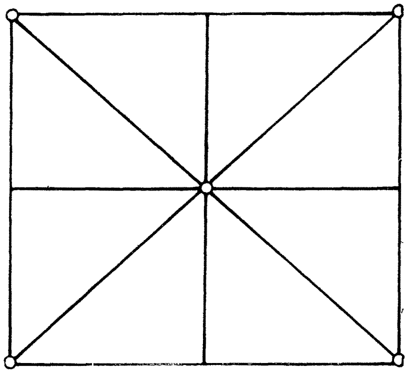

FIG. 5.

This construction works for values of A down to $\sqrt{3}/3 = .577$. Beyond that, the diagonal of the subrectangle becomes longer than the short side A of the original rectangle, so that the above arrangement no longer attains the separation d. Hence for A between .433 and .577, another approach is necessary.

The appropriate argument is illustrated in Figure 6, where $A = .45$. Here, the rectangle is bisected vertically. To distribute the five points between the two halves requires that at least one half, a (1/2)-by-A subrectangle, include three points. Using the known optimal solution for three points from Section 4, the arrangement shown in the figure is obtained in either half. Its mirror image in the other half completes the solution.

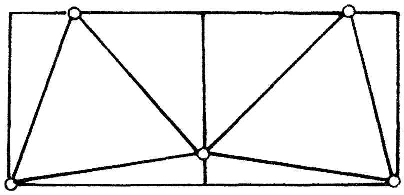

FIG. 6.

There is a delicate point involved in applying the three-point solution to the (1/2)-by-A rectangle in Figure 6. It will be recalled that the longer side of the constraining rectangle is used as the unit of measure. Hence depending on whether $A \geqq 1/2$ or $A \leqq 1/2$, we must view the half of the large rectangle as standing up or lying on its side in applying the three-point solution.

Working out the details will be left to the reader. The resulting orbits are shown in Figure 7. The two bottom points stay fixed. The middle point moves down the vertical bisector. At first it descends half as fast as A. After A passes $\sqrt{3}/3$, the solution point moves down faster, and reaches the middle of the bottom edge when $A = \sqrt{3}/4$. Thereafter it stays there. The two uppermost points move downward. Initially they go down the left and right edges respectively. After A passes $\sqrt{3}/3$, they move inward toward the vertical dividers, reaching those lines when $A = \sqrt{3}/4$. Then they too descend vertically toward the bottom.

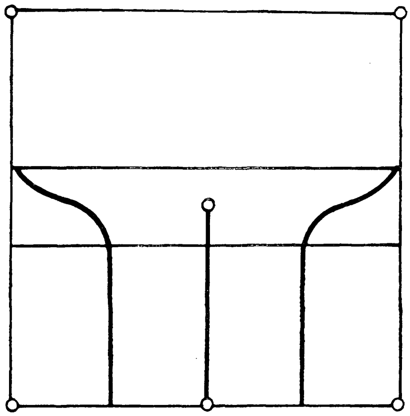

FIG. 7.

7. Four points. With $N = 4$, the orbits have two parts. One of these corresponds to the saw-tooth solution, occuring when $A \leqq \sqrt{3}/3 = .577$. For larger values of A, the solution is obtained as follows. Two diagonally opposite corners are selected, and points located on them. (See Figure 8 for an example, where $A = .75$.) The diagonal is drawn, and its perpendicular bisector constructed. The latter intersects the top and bottom of the rectangle in the two other points of the optimum set. The separation attained is $d = (1/2)\ (1 + A^2)$, as is easily calculated. The four-point arrangement defines a rhombus.

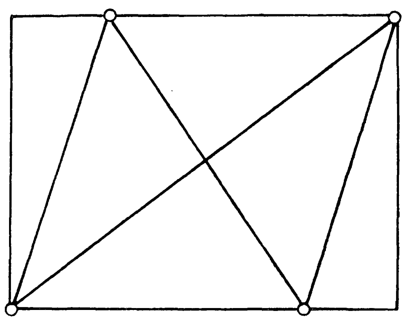

FIG. 8.

To prove this optimal, note first that no point can be in the interior of the rectangle. From Figure 9, we see that every interior point is within distance d of at least two corners. Hence any interior point, if it were to appear, would exclude an entire edge for placement of another point. But this is contrary to the lemma.

Now suppose the point P on the left edge is p units above the corner, $0 \leqq p \leqq A/2$. (This is sufficient, since the cases $A/2 \leqq p \leqq A$ are symmetric about the horizontal bisector.)

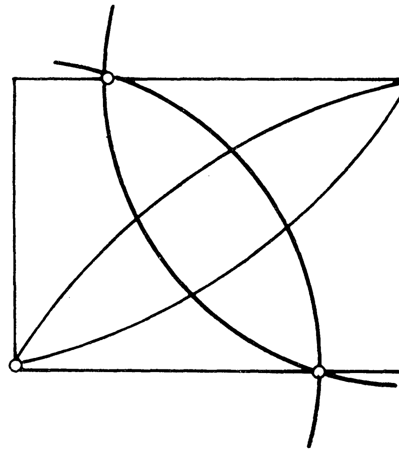

FIG. 9.

In Figure 10, the left-most possible positions for the points Q and R on the top and bottom edges respectively are shown, obtained by striking an arc of radius $d = (1/2)\,(1 + A^2)$ from P as center. Their distances from the left edge are given by $q^2 = d^2 - (A - p)^2$ and $r^2 = d^2 - p^2$ respectively. In turn, to find the highest and lowest position on the right edge that point S can occupy, strike arcs from R and Q respectively with radius d. These two arcs intersect at S'. And $PQRS'$ is a rhombus. So S' is easily located by reflecting the triangle PQR about leg QR. By symmetry, S' is p units below the top of the rectangle (extended if necessary), and $s' = q + r$ units from the left edge.

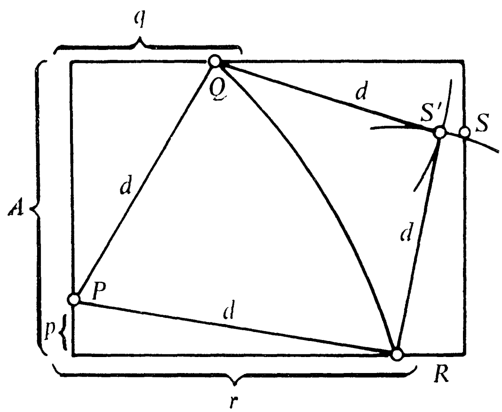

FIG. 10.

We can express s' as a function of p alone. It is a straightforward (although lengthy) exercise in calculus to show that s' is an increasing function of p for $0 \leqq p \leqq A/2$. (See Appendix.) Since $s' = 1$ when $p = 0$, it follows that the point S' is actually outside the rectangle unless $p = 0$; and so the point S can only be placed on the right edge when $p = 0$. This position for S and the corresponding location of P at the lower left corner correspond to the originally asserted optimal solution. This completes the proof.

In Figure 11, the orbits for the four-point case are shown. The lower left point is fixed. The lower right point moves leftward to the vertical trisector and then stays fixed there. The upper right point moves straight down. The upper left point moves down and rightward until it reaches the vertical trisector, and then goes straight down.

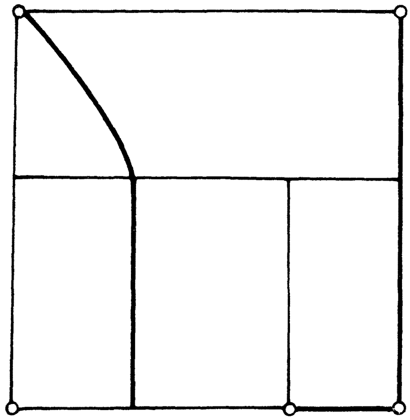

FIG. 11.

8. Additional observations. For larger values of N the problem appears to be much more difficult. The methods that succeeded for the square with $N = 6$, 8, or 9 seem inadequate for the rectangle. With $N = 6$, the optimal solution has only one point in the interior of the square, and the proof makes strong use of that fact. For $N = 8$ or 9, the solutions have central symmetry; and again the solutions use the condition in an essential way. Clearly, both these crutches will ultimately fail in the rectangle, since for small A, the saw-tooth solution occurs, having neither central symmetry nor points in the interior in general. Thus, the orbits of the solution points are known to behave in some new and probably strange fashion.

One conjecture that appeared at first to be reasonable was that every solution in the rectangle, for all A and N, has at least one point in a corner. But in reference **[2]**, Goldberg obtains the best known arrangement in the square with $N = 21$, having no points in any of the corners. This is not conclusive for rejection of the conjecture, since Goldberg does not claim to have proved his arrangement is optimal. But surely the conjecture is substantially weakened by this circumstance. So this author finds himself reluctant to put forth any conjectures about the solutions for more advanced cases — except that they will be hard to find!

9. Appendix. For $0 \leqq p \leqq A/2$, it is desired to verify that the function $s' = q + r$ is monotone increasing in p, where

(1) $$q^2 = [(1/2)(A^2 + 1)]^2 - (A - p)^2, \text{ and}$$

(2) $$r^2 = [(1/2)(A^2 + 1)]^2 - p^2.$$

We show this by proving that $ds'/dp \geqq 0$. It is convenient to study separately dq/dp and dr/dp.

Differentiating (1), we obtain $q\,dq/dp = A - p$, or $dq/dp = (A - p)/q$. The numerator is monotone decreasing in p, and the denominator monotone increasing. Hence dq/dp is monotone decreasing. As p increases from zero, dq/dp starts at $2A/(A^2 - 1)$ and goes down.

In like manner, differentiating (2), we get $dr/dp = -p/r$. Now the fraction p/r (omitting the negative sign) is monotone increasing. For its numerator is increasing and denominator decreasing in p. When the minus sign is adjoined, dr/dp is seen to be also monotone decreasing, starting at zero and immediately going negative as p increases.

The sum $dq/dp + dr/dp = ds'/dp$ is monotone decreasing. To verify that it is nonnegative for all p in the range $0 \leqq p \leqq A/2$, it is merely necessary to verify it for the largest value, $p = A/2$. When this value is substituted we get $ds'/dp|_{p=A/2} = 0$, which is certainly nonnegative. And this completes the proof.

References

1. J. S. Byrnes, The massivity of the square, Amer. Math. Monthly, 78, 4 (1971) 376–378.
2. Michael Goldberg, The packing of equal circles in a square, this MAGAZINE, 43, 1 (1970) 24–30.
3. ———, Packing of 14, 16, 17, and 20 circles in a circle, this MAGAZINE, 44, 3 (1971) 134–139.
4. S. Kravitz, Packing cylinders into cylindrical containers, this MAGAZINE, 40, 1 (1967) 65–71.
5. J. Schaer, On the packing of ten equal circles in a square, this MAGAZINE, 44, 3 (1971) 139–140.
6. ——— and A. Meir, On a geometric extremum problem, Canad. Math. Bull., 8 (1965) 21–27.
7. J. Schaer, The densest packing of 9 circles in a square, Canad. Math. Bull., 8 (1965) 273–277.
8. B. L. Schwartz, Separating points in a square, J. Recreational Math., 3, 4 (1970) 195–204.
9. David Silverman, A max-min problem, J. Recreational Math. (Problem Section), 2 (1969) 161–162.

TWO-DIMENSIONAL GRAPHICAL SOLUTION OF HIGHER-DIMENSIONAL LINEAR PROGRAMMING PROBLEMS

W. P. COOKE, University of Wyoming

In this paper are examples of two-dimensional graphical solutions of linear programming problems involving more than two variables. The technique is mostly a curiosity, and in fact will not solve all linear programming problems. Nevertheless it has value as a teaching device and could be used as a "conjecture-generator" to stimulate undergraduate research. Further, the mathematical level is such that a fascinating unit for a high school class could be based upon the idea.

It will be evident that this graphical technique will not threaten the position of the simplex method as an efficient computational tool. Its principal value resides in its ability to provoke a *thought* in the mind of almost anyone who can graph a straight line.

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6. ——— and A. Meir, On a geometric extremum problem, Canad. Math. Bull., 8 (1965) 21–27.
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It will be evident that this graphical technique will not threaten the position of the simplex method as an efficient computational tool. Its principal value resides in its ability to provoke a *thought* in the mind of almost anyone who can graph a straight line.

The rationale of the technique. There are two basic notions underlying this approach. First, any point which satisfies two inequalities will also satisfy the inequality resulting from their linear combination, provided of course that the sense of that resulting inequality is known. Second, one does not need n dimensions to draw a graph which represents an equation in n variables.

A three-variable example. Consider the problem

$$(1)\qquad \begin{cases} \text{maximize } Z = 2x_1 + 2x_2 + x_3 \\ \text{subject to} \\ \text{(a)}\quad x_1 + x_2 + x_3 \leqq 12 \\ \text{(b)}\quad x_1 + 2x_2 - x_3 \leqq \ 5 \\ \text{(c)}\quad x_1 - \ x_2 + x_3 \leqq \ 2 \\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0. \end{cases}$$

This is of course a standard textbook-type example for which a feasible solution obviously exists while the optimal solution is not so obvious. It requires three ordinary simplex iterations for a solution, or else one could solve it graphically in three dimensions.

We will solve it with two-dimensional graphs. Make the substitution $X = 2x_2 + x_3$. Then the objective function can be written simply as $Z = 2x_1 + X$, but of course the constraints do not explicitly involve X.

Next consider multiplying (b) by the number k and adding to (a), requiring that k be such that the resulting coefficients of x_2 and x_3 be in the ratio $2:1$. Since inequalities rather than equations are involved, we need only to be careful that k is such that the sense of the combined inequality can be determined. In this fashion X can enter explicitly into the constraints.

Thus, blithely ignoring the bother that the resulting feasible region will not be equivalent to the original region in (1), we proceed to combine (a), (b), and (c) into two inequalities, both of which involve only X and x_1. For combining (a) and (b) use k such that

$$(2)\qquad \frac{1+2k}{1-k} = \frac{2}{1}, \text{ or } k = \frac{1}{4},$$

writing

$$(3)\qquad (x_1 + x_2 + x_3 - 12) + (\tfrac{1}{4})(x_1 + 2x_2 - x_3 - 5) \leqq 0.$$

This yields

$$(4)\qquad 5x_1 + 3X \leqq 53.$$

Notice that any feasible point of (1) must satisfy (4) since, in order to be feasible, it must satisfy (a) and (b).

Similarly, to add $k \cdot$(c) to (b) use

(5) $$\frac{2-k}{-1+k} = \frac{2}{1}, \text{ or } k = \frac{4}{3},$$

producing

(6) $$7x_1 + X \leqq 23.$$

Be aware that if either k in (2) or k in (5) had been negative, the sense of the corresponding inequality would have been unknown. That situation may imply that either a different substitution, like $Y = x_1 + x_2$, might have to be used or else, worse yet, no such substitution will work. That is, it is not always possible to achieve the desired two-variable reduced problem.

Having successfully generated (4) and (6) we now consider the reduced problem

(7) $$\begin{cases} \text{maximize } Z_1 = 2x_1 + X \\ \quad \text{subject to} \\ \qquad 5x_1 + 3X \leqq 53 \\ \qquad 7x_1 + X \leqq 23 \\ \qquad x_1 \geqq 0,\ X \geqq 0. \end{cases}$$

Because of the way problem (7) was constructed, any feasible point of (1) is feasible in (7). Thus if (7) has a finite optimal solution, say Z_1^*, then $\max Z \leqq Z_1^*$. Further, it is clear that if there is a feasible solution for (1) such that $Z = Z_1^*$, then that is an optimal solution for (1).

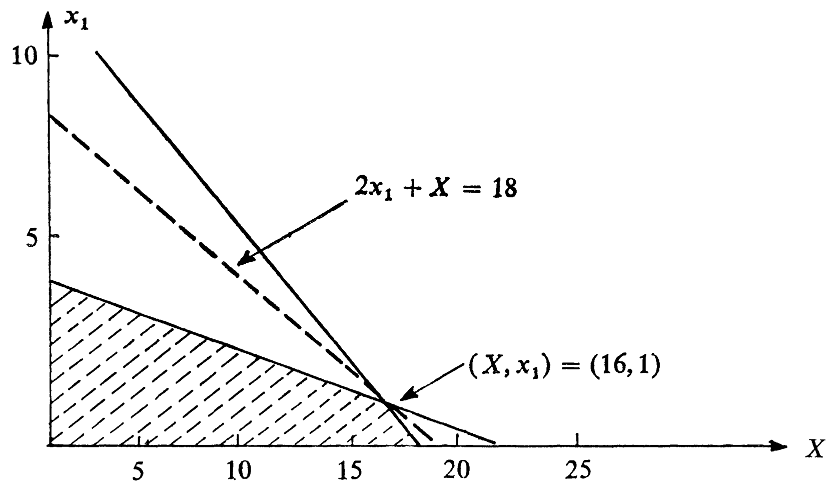

Fig. 1. Solution of Problem (7)

The graphical solution of (7) is suggested by Figure 1. By moving the $2x_1 + X$ contour through the feasible region in the direction of increasing Z_1 we discover $Z_1^* = 18$ at $(X, x_1) = (16, 1)$.

We now ask if there is a feasible point of (1) such that $Z = 18$. If so, it should occur with $x_1 = 1$. Rewrite (1) accordingly and seek a solution of the system

$$(8)\qquad \begin{cases} 2x_2 + x_3 = 16 \\ x_2 + x_3 \leqq 11 \\ 2x_2 - x_3 \leqq 4 \\ -x_2 + x_3 \leqq 1 \\ x_2 \geqq 0,\ x_3 \geqq 0. \end{cases}$$

Because of the equation in system (8), it is easy to reach the condition

$$(9)\qquad 5 \leqq x_2 \leqq 5,$$

so that (8) is satisfied only by $(x_2, x_3) = (5, 6)$. System (8) can also be solved graphically, as shown in Figure 2. This graph is exhibited mainly to reveal that there is no slack in the problem at the optimal solution.

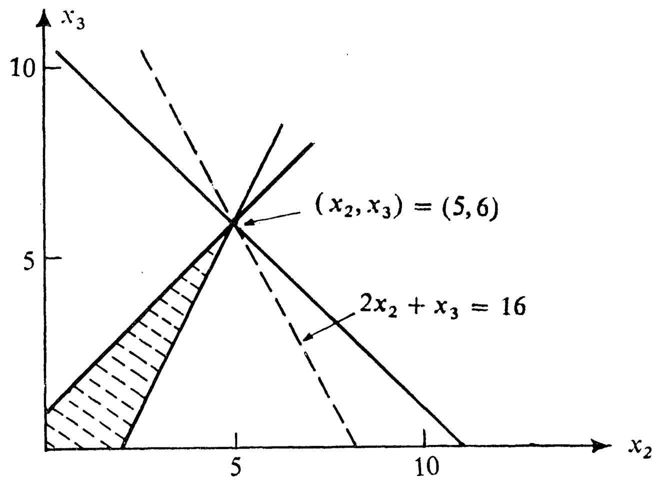

Fig. 2. Solution of System (8)

Since (8) did have a solution, the original problem (1) is now completely solved. Thus

$$(10)\qquad \max Z = 18 \text{ at } (x_1, x_2, x_3) = (1, 5, 6).$$

If one will delete the discussion and merely regard the mechanics of the graphical solution, he will see that the relative difficulty is at least no greater than that required by the simplex method. Further, the graphical solution is by far the more entertaining of the two.

A four-variable example. In this section is exhibited a four-variable, three-constraint problem whose unique optimal solution may be achieved by means of two-dimensional graphs. Consider

$$\text{maximize } Z = 4x_1 + 5x_2 + 7x_3 - x_4$$

$$(11)\quad \begin{cases} \text{subject to} \\ \quad (d)\ \ 2x_1 - x_2 + 3x_3 + 4x_4 \leqq 10 \\ \quad (e)\ \ x_1 + x_2 + x_3 - x_4 \leqq 5 \\ \quad (f)\ \ x_1 + 2x_2 - 2x_3 + 4x_4 \leqq 12 \\ \quad\quad x_1, x_2, x_3, x_4 \geqq 0. \end{cases}$$

Only an outline of steps which lead to the graphical solution is presented, since there are no new concepts. The strategy is the same as is usually employed in solving systems of equations. First we achieve a reduced three-variable problem and then from that a reduced two-variable problem which may be solved graphically. If the reduction has been achieved using judicious substitutions, then we may work back through the problem (graphically) to its solution.

As a first substitution, choose $X = 5x_2 + 7x_3$. Using $(d) + k(e)$ and $(d) + k(f)$ we may obtain

$$(12)\quad \begin{cases} \text{maximize } Z_1 = 4x_1 + X - x_4 \\ \quad \text{subject to} \\ \quad\quad 13x_1 + 2X - 7x_4 \leqq 65 \\ \quad\quad 35x_1 + 2X + 92x_4 \leqq 252 \\ \quad\quad x_1 \geqq 0,\ X \geqq 0,\ x_4 \geqq 0. \end{cases}$$

Continuing, with $Y = X - x_4$ it is easy to achieve

$$(13)\quad \begin{gathered} \text{maximize } Z_2 = 4x_1 + Y \text{subject to } 1397x_1 + 198Y \leqq 7370 \\ x_1 \geqq 0,\ Y \text{ unrestricted in sign.} \end{gathered}$$

The graphical solution of (13) is $(x_1, Y) = (0, 335/9)$, where $Z_2^* = 335/9$. With $x_1 = 0$ the solution of (12) may also be achieved graphically and is $(X, x_4) = (352/9, 17/9)$, where still $Z_1^* = Z_2^* = 335/9$. Continuing with $x_1 = 0$ and $x_4 = 17/9$, problem (11) is now a two-variable problem whose solution is the solution of the system

$$(14)\quad \begin{cases} 5x_2 + 7x_3 = \dfrac{352}{9} \\ -x_2 + 3x_3 \leqq \dfrac{22}{9} \\ x_2 + x_3 \leqq \dfrac{62}{9} \\ 2x_2 - 2x_3 \leqq \dfrac{40}{9} \\ x_2 \geqq 0,\ x_3 \geqq 0. \end{cases}$$

System (14) yields $(x_2, x_3) = (41/9,\ 7/3)$, where $\max Z = 335/9$. Then we have again achieved a feasible solution of (11) so that $\max Z = Z_1^* = Z_2^* = 335/9$, or an optimal solution for (11) is $(x_1, x_2, x_3, x_4) = (0,\ 41/9,\ 7/3,\ 17/9)$.

Comments on possible conjectures. A first likely conjecture that one might make in view of the examples (1) and (11) is that the technique will only work those problems for which all "activity" constraints are satisfied with equality. That such is not the case may be demonstrated by considering the problem

$$(15)\qquad \begin{cases} \text{maximize } Z = 4x_1 + 5x_2 - 3x_3 \\ \text{subject to } x_1 + x_2 + x_3 = 10 \\ x_1 - x_2 \geqq 1 \\ 2x_1 + 3x_2 + x_3 \leqq 20 \\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0. \end{cases}$$

Using the substitution $X = 4x_1 + 5x_2$ it is possible to reach

$$(16)\qquad \begin{cases} \text{maximize } Z_1 = X - 3x_3 \\ \text{subject to} \\ 2X + 9x_3 \leqq 89 \\ X + 3x_3 \leqq 40 \\ X \geqq 0,\ x_3 \geqq 0 \end{cases}$$

which has the solution $(X, x_3) = (40, 0)$, where $Z_1^* = 40$. Then (15) has the solution $(x_1, x_2, x_3) = (10, 0, 0)$ with $\max Z = 40$, but the second constraint is not satisfied with equality.

Secondly, one might suspect that if a final two-variable reduced problem may be achieved and solved graphically, then always $\max Z = Z_j^*$, where j is the number of reductions. Again an invalid conjecture in view of

$$(17)\qquad \begin{cases} \text{maximize } Z = 10x_1 + x_2 + 2x_3 \\ \text{subject to} \\ x_1 + x_2 - 2x_3 \leqq 10 \\ 4x_1 + x_2 + x_3 \leqq 20 \\ x_1, x_2, x_3 \geqq 0. \end{cases}$$

Using $X = 10x_1 + 2x_3$ problem (17) reduces to

$$\text{Maximize } Z_1 = X + x_2$$

$$(18)\quad \begin{cases} \text{subject to} \\ \qquad 9X + 24x_2 \leqq 460 \\ \qquad X \geqq 0,\ x_2 \geqq 0 \end{cases}$$

which has the solution $(X, x_2) = (460/9,\ 0)$, where $Z_1^* = 460/9$. But (17) is solved by $(x_1, x_2, x_3) = (5, 0, 0)$, where $\max Z = 50 < Z_1^* = 460/9$. Notice, however, that if from the solution of (18) we use $x_2 = 0$ in (17), the optimal solution of (17) is gained. Will that strategy always achieve $\max Z$?

One observation that is obviously true follows. If one of the original constraints is an equality, a first reduction is always possible. The more interesting question is: If a first reduction is possible, can a final two-variable reduced problem always be reached?

Of course the larger question is: What is a general *class* of linear programming problems that yield to this technique? Obviously there are classes that do not; for example, those problems for which the number of activity constraints is at least two less than the number of variables.

Conclusion. The graphical solution of two-variable linear programming problems is easy to teach at almost any level of mathematical sophistication. The preceding examples suggest ways to extend that technique with a view to generating conjectures that are more or less difficult to resolve, depending upon the particular level of the student. In any event, the idea is at least entertaining.

EGYPTIAN FRACTION EXPANSIONS

ROBERT COHEN, York College, City University of New York

1. Introduction. It is well known that a positive fraction is expressible as a finite sum of distinct Egyptian fractions in an infinite number of ways (see Beck, Bleicher, Crowe [1] and J. C. Owings [3]). A positive fraction is called Egyptian if its numerator is 1. It is our intention to show a manner of expressing any positive real number as a unique sum of distinct Egyptian fractions. To this end, an algorithm is presented in the proof of the theorem below which terminates in no more than p steps if x is a positive rational number of the form p/q; the algorithm is nonterminating when x is an irrational number. The Egyptian fraction expansion of the real numbers leads us to construct two bijections. The first mapping is an explicit bijection of the set of the positive rational numbers onto the set of positive integers. The second bijection is an explicit map of the set of positive real numbers onto the set of sequences of nonnegative integers.

2. Egyptian fraction expansions. We shall say that a positive real number x

$$
(18) \quad \begin{cases} \text{subject to} \\ \qquad 9X + 24x_2 \leqq 460 \\ \qquad X \geqq 0, \; x_2 \geqq 0 \end{cases}
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EGYPTIAN FRACTION EXPANSIONS

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1. Introduction. It is well known that a positive fraction is expressible as a finite sum of distinct Egyptian fractions in an infinite number of ways (see Beck, Bleicher, Crowe [1] and J. C. Owings [3]). A positive fraction is called Egyptian if its numerator is 1. It is our intention to show a manner of expressing any positive real number as a unique sum of distinct Egyptian fractions. To this end, an algorithm is presented in the proof of the theorem below which terminates in no more than p steps if x is a positive rational number of the form p/q; the algorithm is nonterminating when x is an irrational number. The Egyptian fraction expansion of the real numbers leads us to construct two bijections. The first mapping is an explicit bijection of the set of the positive rational numbers onto the set of positive integers. The second bijection is an explicit map of the set of positive real numbers onto the set of sequences of nonnegative integers.

2. Egyptian fraction expansions. We shall say that a positive real number x

has an *Egyptian fraction expansion* provided

$$x = n_0 + \sum_i \frac{1}{n_1 \cdots n_i} \tag{1}$$

where n_0 is a nonnegative integer and $n_1, n_2, \cdots$ is a nondecreasing sequence of positive integers with $n_1 \geqq 2$, and no term of the sequence appears infinitely often. If x is a nonnegative integer, then $n_0 = [x]$ and the summation in (1) is vacuous. Thus, the numbers $4/73$, $21/13$, e, $\sqrt{2}$ have the following respective expansions:

$$\frac{4}{73} = 0 + \frac{1}{19} + \frac{1}{19 \cdot 25} + \frac{1}{19 \cdot 25 \cdot 37} + \frac{1}{19 \cdot 25 \cdot 37 \cdot 73}$$

$$\frac{21}{13} = 1 + \frac{1}{2} + \frac{1}{2 \cdot 5} + \frac{1}{2 \cdot 5 \cdot 7} + \frac{1}{2 \cdot 5 \cdot 7 \cdot 13}$$

$$e = 2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n!} + \cdots$$

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 5 \cdot 16} + \frac{1}{3 \cdot 5 \cdot 5 \cdot 16 \cdot 18} + \cdots.$$

LEMMA. *For any number y satisfying $0 < y < 1$, there exists a unique integer $n \geqq 2$ and a unique number r such that $1 = ny - r$ and $0 \leqq r < y$. (Of course n is the least integer for which $ny \geqq 1$.)*

Proof. Clear.

THEOREM. *Any positive number x has a unique Egyptian fraction expansion. This expansion terminates if and only if x is rational.*

Proof. Let $n_0 = [x]$, and $y = x - n_0$. We suppose, without loss of generality, that x is not an integer. Then $0 < y < 1$. We set $r_0 = y$, and use the Lemma and induction to choose integers $n_1, n_2, \cdots \geqq 2$ and numbers $r_i \geqq 0$ such that $1 = n_i r_{i-1} - r_i$ and $0 \leqq r_i < r_{i-1}$ for each i. Here we understand that the sequence (r_i) terminates with r_j for some j if $r_0 > r_1 > \cdots > r_{j-1} > r_j = 0$ and this sequence does not terminate if $r_k > 0$ for all $k \geqq 0$. Also $2 \leqq n_1 \leqq n_2 \leqq n_3 \leqq \cdots$ follows from $r_0 > r_1 > r_2 > \cdots$ and our use of the Lemma. Repeated application of $1 = n_i r_{i-1} - r_i$ and induction on k show that for each $k > 0$,

$$y = \sum_{i=1}^{k} \frac{1}{n_1 \cdots n_i} + \frac{r_k}{n_1 \cdots n_i}. \tag{2}$$

From (1) and $0 \leqq r_k < 1 < 2 \leqq n_k$ it follows that

$$y = \sum_i \frac{1}{n_1 \cdots n_i}. \tag{3}$$

Now suppose that x is a rational number. Then $y = p/q$ where p and q are positive integers. From $1 = n_i r_{i-1} - r_i$ it follows that each r_i is an integer divided by q. From this and $r_0 > r_1 > r_2 \cdots \geqq 0$ it follows that the sequence (r_i) terminates with $r_j = 0$ for some $j \leqq p$. Thus the series in (3) terminates. On the other hand, if the series in (3) terminates, y and x are clearly rational.

Now suppose that x is irrational and there is a j and an integer n such that $n_i = n$ for all $i \geqq j$. Then (3) reads

$$y = \sum_{i=1}^{j} \frac{1}{n_1 \cdots n_i} + \frac{1}{n_1 \cdots n_j} \sum_{i>j} \frac{1}{n^{i-j}}.$$

Since $\sum_{i>j} 1/(n^{i-j}) = 1/(n-1)$, it follows that y and x are rational, which is impossible. Thus (n_i) is unbounded if x is irrational.

It remains to prove uniqueness. Suppose that x has the expansions

$$x = n_0 + \sum_i \frac{1}{n_1 \cdots n_i} = m_0 + \sum_i \frac{1}{m_1 \cdots m_i}.$$

Since $2 \leqq n_i$, we have

$$\sum_{i \geqq 1} \frac{1}{n_1 \cdots n_i} \leqq \sum_{i \geqq 1} \frac{1}{2^i} = 1. \tag{4}$$

This last inequality is strict because the series on the left side of (4) terminates if x is rational, and $n_i > 2$ for some i if x is irrational. Likewise $\sum_{i \geqq 1} 1/m_1 \cdots m_i < 1$. So $n_0 = [x]$ and $m_0 = [x]$. Thus $n_0 = m_0$ and

$$\sum_{i \geqq 1} \frac{1}{n_1 \cdots n_i} = \sum_{i \geqq 1} \frac{1}{m_1 \cdots m_i}.$$

Let $w = \sum_{i \geqq 1} \frac{1}{n_1 \cdots n_i}$. Then $n_1 w = 1 + \sum_{i \geqq 2} \frac{1}{n_2 \cdots n_i}$ and

$$1 = n_1 w - \sum_{i \geqq 2} \frac{1}{n_2 \cdots n_i}.$$

But

$$0 \leqq \sum_{i \geqq 2} \frac{1}{n_2 \cdots n_i} \leqq \sum_{i \geqq 1} \frac{1}{n_1 \cdots n_i} < 1$$

so n_1 is the integer determined by w in the Lemma. Likewise m_1 must be the same integer and $n_1 = m_1$. Moreover,

$$\sum_{i \geqq 2} \frac{1}{n_2 \cdots n_i} = \sum_{i \geqq 2} \frac{1}{m_2 \cdots m_i}.$$

Repeated applications of this principle give $n_1 = m_1$, $n_2 = m_2$, $n_3 = m_3, \cdots$. This proves uniqueness.

3. A class of transcendental numbers. We may construct a large class of transcendental numbers by imposing the following restriction on the sequence (n_i): let $n_0 = 0$ and $n_1 \geqq 2$, and let n_{i+1} satisfy the inequality

$$n_{i+1} \geqq (n_1 \cdots n_i)^i n_i + 1 \text{ for } i \geqq 1.$$

The resulting real number with the Egyptian fraction expansion (1) is transcendental if we appeal to Liouville's theorem in [**2**].

Remark. Noting that e has a simple Egyptian fraction expansion, it is natural to ask whether other well known numbers, e.g., π, γ, $\sqrt{2}$ have similarly elementary expansions. Is there an orderly rate of growth of the sequence (n_i) which characterizes an algebraic number when viewed as an Egyptian expansion?

4. Two bijections. A bijection from the positive rationals onto the positive integers will now be constructed. First we construct a bijection from the rationals in the interval $(0,1)$ onto the positive integers. Let y be a rational number such that $0 < y < 1$. We use the algorithm of the theorem to produce the integers $n_1, n_2, \cdots, n_k$. Let $N = 2^{n_1-2} + 2^{n_2-1} + \cdots + 2^{n_k+k-3}$ where $2 \leqq n_1 \leqq n_2 \leqq \cdots \leqq n_k$. The mapping $\phi\colon y \to N$ is invertible, since any positive integer M may be put into binary form $M = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}$ where $0 \leqq m_1 < m_2 < \cdots < m_k$ so that $\phi^{-1}\colon M \to z$ where

$$z = \frac{1}{m_1+2} + \frac{1}{(m_1+2)(m_2+1)} + \cdots + \frac{1}{(m_1+2)(m_2+1)\cdots(m_k-k+3)}.$$

The desired bijection of the positive rationals onto the positive integers is given by $x \mapsto \phi(x/x+1)$.

We now construct a bijection of the set of positive real numbers onto the set of sequences of nonnegative integers. Let x be a positive rational number. Use the algorithm again on $x/(x+1)$ to produce the integers $n_1, n_2, \cdots, n_k$. We define n_0 differently. Let n_0 denote the number of subscripts i for which $1 \leqq i \leqq k-1$ and $n_i = n_k$. We associate x with the sequence n_0, $n_1 - 2$, $n_2 - n_1$, $n_3 - n_2, \cdots$, $n_k - n_{k-1}, 0, 0, 0, \cdots$. This provides an explicit bijection of the set of positive rational numbers onto the set of sequences of nonnegative integers with finitely many positive terms.

On the other hand, let x be a positive irrational number. We use the algorithm of the theorem on x to determine the integers $n_0, n_1, n_2, n_3, \cdots$. We associate x with the sequence n_0, $n_1 - 2$, $n_2 - n_1$, $n_3 - n_2, \cdots$. This provides an explicit bijection of the set of positive irrational numbers onto the set of sequences of nonnegative integers with infinitely many positive terms.

Finally, we have an explicit bijection of the set of positive real numbers onto the set of sequences of nonnegative integers such that irrational numbers correspond to sequences with infinitely many positive terms and rational numbers correspond to sequences with finitely many positive terms. If the sequence $(m_0, m_1, m_2 \cdots)$ has infinitely many positive terms then it corresponds to the number

$$x = m_0 + \frac{1}{2+m_1} + \frac{1}{(2+m_1)(2+m_1+m_2)} + \frac{1}{(2+m_1)(2+m_1+m_2)(2+m_1+m_2+m_3)} + \cdots$$

which we have shown to be irrational. If, on the other hand, m_j is the last positive term, the sequence corresponds to the number $x = y/(1-y)$ where

$$y = \frac{1}{2+m_1} + \frac{1}{(2+m_1)(2+m_1+m_2)} + \frac{1}{(2+m_1)\cdots(2+m_1+\cdots+m_{j-1})}$$
$$+ \frac{1}{(2+m_1)\cdots(2+m_1+\cdots+m_j)}\left[\sum_{i=0}^{m_0} \frac{1}{(2+m_1+\cdots+m_j)^i}\right]$$

which is of course rational.

The author wishes to express his thanks to the referee who suggested this second mapping and for his helpful comments in organizing this paper.

References

1. A. Beck, M. Bleicher, D. Crowe, Excursions into Mathematics, Worth Publishers, New York, 1969.
2. A. Ya. Khinchin, Continued Fractions, University of Chicago Press, Chicago, 1964, p. 47.
3. J. C. Owings, Another proof of the Egyptian fraction theorem, Amer. Math. Monthly, 75 (1968) 777–778.

THE PERMANENT FUNCTION AND THE PROBLEM OF MONTMORT

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The problem of Montmort is concerned with D_n, the number of distinct ways n objects can be arranged with no object remaining fixed. Several methods [**1**, pages 23 and 31] have been employed to obtain

$$D_n = n! \sum_{i=0}^{n} (-1)^i \frac{1}{i!}. \tag{1}$$

It is interesting to the author that H. J. Ryser [**1**, page 28], showed

$$D_n = \mathrm{per}(S_n), \tag{2}$$

where $S_n = (s_{ij})$ is an n-square (0,1) matrix with $s_{ij} = 1 - \delta_{ij}$ (δ_{ij} denotes the Kronecker delta). Thus, S_n is the square matrix with zeros on the main diagonal, and 1's in all other positions. For example,

$$S_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

In this paper, we shall assume (2), but not (1), and develop a new, algebraic method for evaluation of $\mathrm{per}(S_n)$, based on two properties of the permanent function.

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$$+ \frac{1}{(2+m_1)\cdots(2+m_1+\cdots+m_j)}\left[\sum_{i=0}^{m_0} \frac{1}{(2+m_1+\cdots+m_j)^i}\right]$$

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References

1. A. Beck, M. Bleicher, D. Crowe, Excursions into Mathematics, Worth Publishers, New York, 1969.

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In this paper, we shall assume (2), but not (1), and develop a new, algebraic method for evaluation of $\text{per}(S_n)$, based on two properties of the permanent function.

Recall that per A is defined as

$$\operatorname{per} A = \sum a_{1i_1} a_{2i_2} \cdots a_{ni_n},$$

where the sum extends over all permutations $(i_1, i_2, \cdots, i_n)$ and $A = (a_{ij})$ denotes an n-square matrix. The permanent resembles the familiar determinant function, $\det A$; they differ in that the terms in the $\operatorname{per} A$ summation all have positive sign, whereas those in $\det A$ have signs depending on the permutation $(i_1, i_2, \cdots, i_n)$. Two properties of the permanent function follow immediately from the definition:

(1) *if B is a matrix obtained from A by the interchange of rows (or columns) of A, then* $\operatorname{per} B = \operatorname{per} A$;

(2) *if $A = (a_{ij})$ is an n-square matrix, then* $\operatorname{per} A = \sum^n{}_{j=1} a_{ij}$ per $A(i,j)$ $(i = 1, 2, \cdots, n)$, *where $A(i,j)$ is the matrix obtained from A by deleting the* ith *row and* jth *column.*

Let T_n be defined as an n-square matrix (t_{ij}), where $t_{ii} = 0$ for $i = 2, \cdots, n$, and $t_{ij} = 1$ otherwise. Thus T_n and S_n are identical, except in the upper left corner entry, the $(1,1)$ position. Here T_n has a 1, while S_n has a zero. So, for example

$$T_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Observe, by using property (2), with $i = 1$ (that is, expanding the permanent by the first row),

$$\operatorname{per} S_4 = \operatorname{per} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= 0 \cdot \operatorname{per} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \operatorname{per} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$+ \operatorname{per} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \operatorname{per} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

By interchanging the first and second rows of $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, we obtain $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Also, by interchanging the first and the third rows of $\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}$, we obtain $\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}$ and then interchanging the second and third rows of this matrix we have $\begin{bmatrix}1&1&1\\1&0&1\\1&1&0\end{bmatrix}$.

Thus, using property (1),we have

$$\text{per}\begin{bmatrix}1&0&1\\1&1&1\\1&1&0\end{bmatrix} = \text{per}\begin{bmatrix}1&1&1\\1&0&1\\1&1&0\end{bmatrix} = \text{per } T_3$$

$$\text{and per}\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix} = \text{per}\begin{bmatrix}1&1&1\\1&0&1\\1&1&0\end{bmatrix} = \text{per } T_3.$$

Hence, per $S_4 = 3$ per $T_3 = (4-1)$ per T_3. This leads us to our first lemma.

LEMMA 1. *If S_n and T_n are defined as above, then* per $S_n = (n-1)$ per T_{n-1}.

Proof. Using property (2), we have

$$\text{per}\, S_n = \sum_{j=2}^{n} \text{per}[S_n(1,j)].$$

But the $(j-1)$th row of $S_n(1,j)$ consists only of 1's and each remaining row consists of 1's and exactly one zero, which is not in the same column as any of the other zeros of $S_n(1,j)$. Hence, as demonstrated in the preceding example, by permuting rows of $S_n(1,j)$ and using property (1), for $j \neq 1$, we have

$$\text{per}[S_n(1,j)] = \text{per}\, T_{n-1}.$$

Thus, per $S_n = \sum_{j=2}^{n}$ per $T_{n-1} = (n-1)$ per T_{n-1}.

LEMMA 2. per T_{n-1} = per $S_{n-2} + (n-2)$ per T_{n-2}.

Proof. From property (2),

$$\text{per}\, T_{n-1} = \sum_{j=1}^{n-1} \text{per}[T_{n-1}(1,j)].$$

But per $[T_{n-1}(1,1)]$ = per S_{n-2}. Since each row of T_{n-1} is identical to the corresponding row of S_{n-1} except for the first, we have $T_{n-1}(1,j) = S_{n-1}(1,j)$. Also, in the proof of Lemma 2 it was shown that per $S_{n-1}(1,j)$ = per T_{n-2}. Thus, per $T_{n-1}(1,j)$ = per T_{n-2}. Therefore, per T_{n-1} = per S_{n-2} + $\sum_{j=2}^{n-1}$ per T_{n-2} = per $S_{n-2} + (n-2)$ per T_{n-2}.

LEMMA 3. per $S_n = (n-1)$ [per S_{n-1} + per S_{n-2}].

Proof. From Lemma 1 and Lemma 2, we have

$$\begin{aligned}\operatorname{per} S_n &= (n-1)\operatorname{per}(T_{n-1})\\ &= (n-1)\left[\operatorname{per}(S_{n-2}) + (n-2)\operatorname{per}(T_{n-2})\right].\end{aligned}$$

Now using Lemma 1 again, we have

$$\operatorname{per} T_{n-2} = \frac{1}{n-2}\operatorname{per}(S_{n-1}).$$

Substituting this in the above expression, we obtain the lemma.

THEOREM. $\operatorname{per} S_n = n!\ \sum_{i=0}^{n}(-1)^i\ \frac{1}{i!}$.

Proof. We prove this by mathematical induction. For $n = 1$, $\operatorname{per} S_1 = \operatorname{per}(0) = 0$ and $1!\ \sum_{i=0}^{1}(-1)^i/i! = 1 + (-1) = 0$. Now, let k denote a positive integer and assume the theorem holds for all $n \leqq k$. Thus, using Lemma 3, we have

$$\begin{aligned}\operatorname{per} S_{k+1} &= ((k+1)-1)(\operatorname{per} S_k + \operatorname{per} S_{k-1})\\ &= k\left[k!\ \sum_{i=0}^{k}\frac{(-1)^i}{i!} + (k-1)!\ \sum_{i=0}^{k-1}\frac{(-1)^i}{i!}\right]\\ &= k!\left[k\ \sum_{i=0}^{k}\frac{(-1)^i}{i!} + \sum_{i=0}^{k-1}\frac{(-1)^i}{i!}\right]\\ &= k!\left[k\ \sum_{i=0}^{k}\frac{(-1)^i}{i!} + \frac{(-1)^{k+1}}{k!} + \frac{(-1)^k}{k!} + \sum_{i=0}^{k-1}\frac{(-1)^i}{i!}\right]\\ &= k!\left[k\ \sum_{i=0}^{k}\frac{(-1)^i}{i!} + \frac{(-1)^{k+1}}{k!} + \sum_{i=0}^{k}\frac{(-1)^i}{i!}\right]\\ &= k!\left[(k+1)\ \sum_{i=0}^{k}\frac{(-1)^i}{i!} + \frac{(-1)^{k+1}}{k!}\right]\\ &= k!(k+1)\left[\sum_{i=0}^{k}\frac{(-1)^i}{i!} + \frac{(-1)^{k+1}}{(k+1)k!}\right]\\ &= (k+1)!\ \sum_{i=0}^{k+1}\frac{(-1)^i}{i!}.\end{aligned}$$

This proves the theorem.

We remark that Ryser [**1**, page 31] derived the recurrence relation $D_n = (n-1)(D_{n-1} + D_{n-2})$ which is our Lemma 3 by combinatorial arguments and asserted that induction could be used to obtain (1).

The author wishes to thank the referee for his helpful suggestions which improved very much the clarity of the earlier version of this paper.

Reference

1. H. J. Ryser, Combinatorial Mathematics, Chaps. 2 and 3, Carus Mathematical Monograph No. 14, Mathematical Association of America, 1963.

ALMOST PERFECT NUMBERS

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1. Introduction. We will denote the sum of all of the factors of a positive integer M by $\sigma(M)$. A number M is perfect if the sum of its factors, counting 1 but not counting M, is equal to M. We shall call M *perfect-minus-one* (PM1) if the sum of its factors, counting for the sake of symmetry neither 1 nor M, is equal to M. We put

$$F(M) = 2M - \sigma(M), \tag{1}$$

where $F(M)$ is the *factor difference* of M. If $F(M) = 0$, M is perfect, if $F(M) = -1$, M is PM1. Evidently nothing is known about PM1 numbers. Our purpose is to provide some information, though we do not know whether such a number exists.

2. PM1 numbers. Suppose that $K = 2^n p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s}$, where the numbers p_i are odd primes. The sum of all the factors of K is given by

$$\sigma(K) = (2^{n+1} - 1)\frac{p_1^{n_1+1} - 1}{p_1 - 1}\,\frac{p_2^{n_2+1} - 1}{p_2 - 1} \cdots \frac{p_s^{n_s+1} - 1}{p_s - 1}.$$

Our principal result is the following

THEOREM. *If K is* PM1, *then K is an odd square.*

Proof. If K is PM1, then

$$\sigma(K) = 2K + 1.$$

Since the right hand side is odd, each factor in our representation of $\sigma(K)$ must be odd. Now the first factor is always odd, and $(p^{m+1} - 1)/(p - 1) = 1 + p + p^2 + \cdots + p^m$, which if p is odd is the sum of $(m + 1)$ odd numbers. Hence all of the exponents n_i must be even, and K has the form

$$K = 2^n p_1^{2n_1} p_2^{2n_2} \cdots p_s^{2n_s} = 2^n M^2.$$

Sustituting $K = 2^n M^2$ into (1) and using $\sigma(K) = (2^{n+1} - 1)\sigma(M^2)$ we find

$$\sigma(M^2) - M^2 = \frac{M^2 + 1}{2^{n+1} - 1} = \text{integer}$$

and

$$M^2 \equiv (-1) \bmod (2^{n+1} - 1).$$

But (-1) is a quadratic residue of primes of the form $4k + 1$ and a nonresidue of primes of the form $4k + 3$ [3]. Further, $2^{n+1} - 1$ always has a prime factor of the form $4k + 3$ for it is itself of the form $4k + 3$ $(n > 0)$. Thus the congruence has no solution and the theorem is proved.

We now know that if N is PM1 it has the form

$$N = p_1^{2n_1} p_2^{2n_2} \cdots p_s^{2n_s}, \qquad p_i \text{ odd};$$

hereafter we shall denote by N a number exactly of this form. We give below some necessary conditions for N to be PM1.

PROPOSITION 1. *If $F(L) \leqq 0$ and $M = kL$, then $F(M) < F(L)$. In particular, if $F(L) < 0$ then no multiple of L can be* PM1.

Proof. Case 1. $M = Lp^n$ where p is not a factor of L. We have $\sigma(M) = \sigma(L)(p^{n+1} - 1)/(p - 1)$, and using eq. (1) we find

$$F(L) - F(M) = [2L - pF(L)]\,\frac{p^n - 1}{p - 1}, \tag{2}$$

from which the conclusion follows.

Case II. $M = Lp_i^n$ where p_i^m is the higest power of p_i which divides L. Here $\sigma(M) = \sigma(L)(p_i^{m+n+1} - 1)/(p_i^{m+1} - 1)$ and we find

$$F(L) - F(M) = [2L - p_i^{m+1}F(L)]\,\frac{p_i^n - 1}{p_i^{m+1} - 1}.$$

The right hand side is positive if $F(L) \leqq 0$ so the proof is complete. We find for example that $F(3^2 . 5^2 . 7^2) = -919$, and therefore no multiple of this number is PM1. This is the smallest odd square with a negative factor difference.

PROPOSITION 2. *If N is* PM1 *then*

$$\left(\frac{p_1}{p_1 - 1} \cdots \frac{p_s}{p_s - 1}\right)\left(\frac{p_1^3 - 1}{p_1^3} \cdots \frac{p_s^3 - 1}{p_s^3}\right) \leqq 2 + \frac{1}{N} < \left(\frac{p_1}{p_1 - 1} \cdots \frac{p_s}{p_s - 1}\right).$$

Proof. We again start with (1), and divide by N to obtain

$$\frac{p_1^{2n_1+1} - 1}{p_1^{2n_1}(p_1 - 1)} \cdots \frac{p_s^{2n_s+1} - 1}{p_s^{2n_s}(p_s - 1)} = 2 - \frac{F(N)}{N}.$$

The left hand side is a monotone increasing function of $n_1, n_2, \cdots, n_s$. Its minimum occurs when $n_1 = n_2 = \cdots = n_s = 1$, and it is bounded above by its limit as $n_i \to \infty$ $(1 = 1, 2, \cdots, s)$. Therefore

$$\frac{p_1^3 - 1}{p_1^2(p_1 - 1)} \cdots \frac{p_s^3 - 1}{p_s^2(p_s - 1)} \leqq 2 - \frac{F(N)}{N} < \frac{p_1}{p_1 - 1} \cdots \frac{p_s}{p_s - 1}.$$

If N is PM1 then $F(N) = -1$ and the proof is complete.

Proposition 2 gives a fairly stringent necessary condition for N to be PM1. It says in particular that $(p_1/p_1 - 1)(p_2/p_2 - 1)\cdots(p_s/p_s - 1) > 2$ if N is PM1. Thus is 3 not a factor of N and we satisfy this condition using consecutive primes after 3 (which gives the smallest N and the least number of primes) we find that every prime from 5 to 23 must be a factor of N. Therefore, if 3 is not a factor of N and N is PM1, the number of distinct prime factors of N is at least 7 and

$$N > (5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23)^2 = 1{,}382{,}511{,}906{,}801{,}025.$$

More general arguments of this type are given in [1], with reference to perfect numbers.

PROPOSITION 3. *If* $M = Np^{2n}$, $p \nmid N$ *and* M *is* PM1, *then*

$$\frac{2N}{F(N)} - \frac{1}{p} \leqq p < \frac{2N}{FN}.$$

In particular,

$$\frac{2N}{F(N)} - \frac{1}{3} \leqq p < \frac{2N}{F(N)}.$$

Proof. From Case I of Proposition 1 we have

$$F(N) - F(M) = [2N - pF(N)]\, \frac{p^{2n} - 1}{p - 1}. \tag{2}$$

If M is PM1, then $F(N) \geqq 0$ and $F(M) = -1$. It follows that $2N - pF(N) > 0$, so the right hand inequality is proved.

To prove the left hand inequality we put $F(M) = -1$, $n = 1$, in the above equation and have

$$F(N) + 1 \geqq [2N - pF(N)]\,(p + 1).$$

Since $2N - pF(N)$ is an integer greater than zero, we see that $F(N) \geqq p$. Further, rearranging yields

$$p \geqq \frac{2N}{F(N)} - \frac{1}{p+1} - \frac{1}{F(N)\,(p+1)},$$

and using $F(N) \geqq p$ gives

$$p \geqq \frac{2N}{F(N)} - \frac{1}{p}.$$

This proposition says that if $F(N) > 0$ and $M = Np^{2n}$, there is at most one candidate for the prime p if M is to be PM1. It provides a convenient way of seeking PM1 numbers.

We carried out a search for PM1 numbers by computing $F(K^2)$ for each odd number K which is divisible by 3 up to $K = 9999$. This was done on a large computer in 22.72 seconds at a cost of $5.25. We are indebted to Mr. Lynn Trowbridge for the programming. The results indicate that the likelihood of finding a PM1 number by routine calculation is small. In all but one case the number of digits of K^2 minus the number of digits of $F(K^2)$ was 0, 1, or 2. The exception was $K = 3 \cdot 7 \cdot 11 \cdot 13$, where $F(K^2) = F(9018009) = 819$. This was the smallest factor difference found for any $K > 45$.

3. Perfect and PP1 numbers. We shall call a number K *perfect-plus-one* (PP1) if $F(K) = 1$. Every power of 2 is PP1 but we do not know any other examples. The proof for PM1 numbers shows that whenever $F(M)$ is odd, M has the form $2^n N$, where N is an odd square. Also, it is easy to see that if $M = K(2K - 1)$ where K is PP1 and $(2K - 1)$ is prime, then M is perfect. The even perfect numbers are examples of this. It follows that each even PP1 number M such that $2M - 1$ is prime is a power of 2.

Proposition 3 is true for PP1 numbers; the proof is the same. For odd perfect numbers we find: if $M = Np^n$ and M is perfect, then$(2N/F(N)) - (1/(p+1)) \leqq p < (2N/F(N))$, the equality occurring when $n = 1$. Using this proposition it is easy to see, given N and $F(N)$, whether any number of the form $M = Np^n$ $(p \neq p_i)$ is PP1, perfect, or PM1. Failing this, one can obtain numbers with relatively small factor differences.

In the above example $K = 3 \cdot 7 \cdot 11 \cdot 13$ and $2K^2/F(K^2) = 22022$ exactly. The nearest prime is 22027, so no number of the form K^2p^n is PP1, perfect, or PM1. We do find that if $M = 22027\,K^2$

$$F(M) = F(198,639,683,243) = 4914.$$

Though this is hardly a small factor difference, we note that $2M$ and $\sigma(M)$ agree to eight decimal places.

References

1. Karl Norton, Remarks on the number of factors of an odd perfect number, Acta Arith., 6 (1961) 365.

2. P. J. McCarthy, Odd perfect numbers, Scripta Math., 23 (1957) 43.

3. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Clarendon Press, Oxford, 1960.

PROPERTIES OF A GAME BASED ON EUCLID'S ALGORITHM

EDWARD L. SPITZNAGEL, JR., Washington University

1. Introduction. In a recent paper [3] Cole and Davie have described a remarkable little game based on the Euclidean algorithm. Appropriately, they have given it the name Euclid. As they point out, Euclid furnishes a simple example of a game with an explicit winning strategy. In [3] the authors were primarily concerned with the exposition of that winning strategy and did not mention any of the other intriguing features of Euclid. In this paper we give an account of some of these other features.

We first give a brief account of the game and winning strategy, thus making this paper self-contained, and then pass on to study the other properties.

2. Rules and strategy. We paraphrase the rules of Euclid from [3]: Let (p, q) be a pair of positive numbers satisfying $p > q$ and let A, B be two players. Each player in turn must move. A move consists of replacing the larger of two numbers given him by any nonnegative number obtained by subtracting a positive multiple where N is an odd square. Also, it is easy to see that if $M = K(2K - 1)$ where K is PP1 and $(2K - 1)$ is prime, then M is perfect. The even perfect numbers are examples of this. It follows that each even PP1 number M such that $2M - 1$ is prime is a power of 2.

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2. Rules and strategy. We paraphrase the rules of Euclid from [3]: Let (p, q) be a pair of positive numbers satisfying $p > q$ and let A, B be two players. Each player in turn must move. A move consists of replacing the larger of two numbers given him by any nonnegative number obtained by subtracting a positive multiple of the smaller number from the larger number. The winner of the game is the person who obtains 0 for the (new) smaller number. For example, starting with the pair (51, 30), the successive moves in the game could be:

	or	
(51, 30)		(51, 30)
A: (30, 21)		A: (30, 21)
B: (21, 9)		B: (21, 9)
A: (9, 3)		A: (12, 9)
B: (3, 0) *B* wins		B: (9, 3)
		A: (3, 0) *A* wins.

The strategy can be summarized as follows:

DEFINITION. *Let*

$$c = (\sqrt{5}-1)/2 \doteq .618.$$

(This is the golden section.) Let (p,q), $p > q$ *be a pair occurring in the game of Euclid. Call* (p,q) *a safe position if* $q/p > c$. *Otherwise call* (p,q) *an unsafe position.*

PROPOSITION 1. *A player moving from an unsafe position is always capable of moving to a safe position.*

PROPOSITION 2. *A player moving from a safe position can make just one move and that move will always be to an unsafe position.*

Combining the above two results, we see that a player A who once is given an unsafe position from which to move can ensure that in all future moves he, A, will always be able to move to safe positions and that the other player, B will always be forced to move to unsafe positions. Since the larger number p strictly decreases as the game progresses and since $p \neq q$ at the start, the game must eventually pass through the unsafe position

$$(1) \qquad (kq, q) \quad k \text{ an integer } > 1.$$

Therefore the winning strategy is for A always to move to safe positions until B is forced to move to position (1). Then A can produce 0 on his next move and win.

3. Strategy without the golden section. Can a person be taught the winning strategy without his knowing the golden section c? The answer is yes. A close look at the game reveals the following result:

PROPOSITION 3. *Let* $p > 2q$ *and let* p/q *be nonintegral. Then the person* A *about to move can move to at least two positions, but exactly one of all his possible moves will yield a safe position. If* p_0 *denotes the remainder upon division of* p *by* q, *and* $p_1 = p_0 + q$, *then the safe position is whichever of*

$$(p_1, q) \qquad\qquad (q, p_0)$$

has the larger ratio of second entry to first entry.

Proof. Since $q/p < 1/2 < c$, the position from which A moves is unsafe. By Proposition 1, therefore, A is able to move to a safe position. The various positions to which A can move are (q, p_0), (p_1, q), and all positions of the form $(p_0 + kq, q)$, $k = 2, 3, \cdots$, such that $p_0 + kq < p$. Since for $k \geqq 2$, $q/(p_0 + kq) < q/(kq) = 1/k < c$, the only two positions that can possibly be safe are (q, p_0) and (p_1, q). We know at least one must be safe, and we will now show that both cannot be. Suppose (q, p_0) is safe, so that $p_0/q > c$. Then

$$\frac{p_1}{q} = \frac{p_0 + q}{q} = \frac{p_0}{q} + 1 > c + 1 = \frac{1}{c}.$$

Therefore, $q/p_1 < c$, so that (p_1, q) is unsafe. We have now shown that exactly one of (q, p_0), (p_1, q) is safe. That is, exactly one of these pairs will have ratio of second entry to first entry greater than c. Therefore, whichever pair has the larger ratio of second entry to first entry is the unique safe position. This completes the proof.

In Euclid the only two types of position (p, q) at which one has a choice of moves are those in which

(1) p/q is an integer $\geqq 2$, or

(2) $p > 2q$ and p/q is not an integer.

The latter case is the hypothesis of Proposition 3.

Therefore, the strategy can be rephrased as follows, with no mention of the golden ratio c: If there is a choice of moves and p/q is an integer, return 0 and win. If there is a choice of moves and p/q is not an integer, compute

$$p_0 = \text{remainder after } p \text{ is divided by } q$$

$$p_1 = p_0 + q$$

and move to whichever of (p_1, q), (q, p_0) has the larger ratio of second to first entry.

Of course, one need not actually compute the ratios to determine which is the larger. If $p_0 p_1 < q^2$, then q/p_1 is the larger ratio, so (p_1, q) is the safe position; otherwise (q, p_0) is the safe position. For example, when presented with the pair (70, 11), one would compute $p_0 = 4$, $p_1 = 15$. Then since $4 \cdot 15 < 11^2$, he would make the move (15, 11) rather than (11,4).

4. Relation to Fibonacci series. Once the strategy has been mastered by a person, he is then likely to turn his interest to those parts of the game in which neither player must know the strategy—that is, those parts consisting of a string of moves in which neither player has any choice. In such a string of moves, each player in turn is presented with a pair of numbers $p > q$ such that $p < 2q$ and so can only move to the position $(q, p - q)$.

Of course, eventually such a string must end, with one or the other player being presented with a pair of numbers (p,q) with $p \geqq 2q$. In case such a pair arises only once in the game, it occurs right at the end and is of the form (1). In this case, the various numbers in the ordered pairs can be found by working backward. They are:

$$q,\ kq,\ (k+1)q,\ (2k+1)q,\ (3k+2)q,\ (5k+3)q, \cdots$$

The above sequence of numbers is q times a Fibonacci sequence. The fact that Fibonacci sequences are closely connected to the game is not surprising, since their relations to the Euclidean algorithm [**4**] and to the golden section [**1,7**] are well known.

Now suppose we have a string of forced moves that ends in a position (p_0, q_0), $p_0 \geqq 2q_0$, not necessarily of the form (1). Working backward through the string, we again see a Fibonacci sequence, with the ith move prior to (p_0, q_0) given recursively by

$$(p_i, q_i) = (p_{i-1} + q_{i-1}, p_{i-1}). \tag{2}$$

Then we have the following result, similar to the well-known result [**2, 5, 7**] on the convergents of a continued fraction.

THEOREM. *Let p_i and q_i be defined recursively by* (2), *starting with the pair* (p_0, q_0). *Then*

(i) q_{2k}/p_{2k} *is an increasing function of* k, $k = 0,1,2,\cdots$

(ii) q_{2k+1}/p_{2k+1} *is a decreasing function of* k, $k = 0,1,2,\cdots$.

Proof. By (2) we have

$$\begin{aligned} q_{i+1}p_i - p_{i+1}q_i &= p_i(p_{i-1} + q_{i-1}) - (p_i + q_i)p_{i-1} \\ &= -(q_ip_{i-1} - p_iq_{i-1}). \end{aligned}$$

Applying this result i times, we obtain

$$q_{i+1}p_i - p_{i+1}q_i = (-1)^i(q_1p_0 - p_1q_0). \tag{3}$$

Let $d = q_1p_0 - p_1q_0$. Then from (3) we obtain

$$\frac{q_{i+1}}{p_{i+1}} - \frac{q_i}{p_i} = (-1)^i d \frac{1}{p_{i+1}p_i}$$

$$\frac{q_{i+2}}{p_{i+2}} - \frac{q_{i+1}}{p_{i+1}} = (-1)^{i+1} d \frac{1}{p_{i+2}p_{i+1}}$$

and therefore

$$\frac{q_{i+2}}{p_{i+2}} - \frac{q_i}{p_i} = (-1)^i d\left(\frac{1}{p_{i+1}p_i} - \frac{1}{p_{i+2}p_{i+1}}\right).$$

Since $p_{i+2} > p_i$,

$$\frac{1}{p_{i+1}p_i} - \frac{1}{p_{i+2}p_{i+1}} > 0$$

and since $p_0 \geqq 2q_0$,

$$\begin{aligned} d &= q_1p_0 - p_1q_0 \\ &= p_0^2 - (p_0 + q_0)q_0 \\ &\geqq 2p_0q_0 - p_0q_0 - q_0^2 = q_0(p_0 - q_0) \\ &> 0. \end{aligned}$$

Therefore

$$\frac{q_{i+2}}{p_{i+2}} > \frac{q_i}{p_i}$$

for i even, while the reverse inequality is true for i odd. This completes the proof.

Returning to the direction in which the game is played, we can thus say the following about any string of forced moves: The player A who is once presented with two numbers of ratio $q/p < c$ finds thereafter that the ratios of the numbers presented him steadily fall until finally he is given a pair with ratio $\leqq 1/2$ and so is enabled to make a decision. If player A always makes his decisions according to the strategy, then player B finds himself completely trapped. During the strings of forced moves B finds the ratios q/p given to him steadily rising to 1 while the ratios he must give to A steadily decrease until one falls below 1/2. Then A chooses a move that may lower the ratio of the numbers he gives to B, but as soon as the next string of forced moves begins, B finds once again that the ratios q/p given him go the wrong way, increasing toward 1, so that he never has an opportunity to make a decision.

Compared to Nim, then, the workings of the strategy in Euclid lie much closer to the surface. In Nim, the opponent of a player following the strategy usually has more than one move open to him at every stage of the game except the last. He therefore does not get such an obvious warning that he is doomed to lose. In Euclid, on the other hand, the opponent of someone following the strategy is likely to notice his moves are being forced every step of the way, and from this observation it might be possible for him to determine what the strategy must be.

5. Probability of starting with a safe position. Now that the strategy of Euclid has been explicated, there arises the question, What is the probability that an arbitrarily chosen starting position is unsafe?

The following would seem to be a reasonable interpretation of an arbitrarily chosen starting position: Both numbers are chosen from the set S of positive integers less than or equal to some positive integer n. The first number x is chosen with uniform probability from S, and the second is chosen with uniform probability from $S \sim \{x\}$. After the two numbers are chosen, they are labeled p and q so that $p > q$.

Under these assumptions, standard probability arguments, as presented in [6] show that q can be considered as drawn uniformly from the set $\{1,2,\cdots,p-1\}$. Given p, we have

$$P(q/p < c) = P(q < cp)$$
$$= [cp]/(p-1)$$

where the square brackets denote the integral part. For large p, this fraction approaches c, so for large n, the probability of an arbitrarily chosen starting position being unsafe approaches c. That is, the first to move, player A, is the more likely to have a winning strategy available.

References

1. W. W. R. Ball, Mathematical Recreations and Essays, Chapter 2, Macmillan, New York, 1960.
2. G. Chrystal, Textbook of Algebra, Chapter 32, Dover, New York, 1961.
3. A. J. Cole and A. J. T. Davie, A game based on the Euclidean algorithm and a winning strategy for it, Math. Gaz., 53 (1969) 354–357.
4. J. D. Dixon, A simple estimate for the number of steps in the Euclidean algorithm, Amer. Math. Monthly, 78 (1971) 374–376.
5. A. Ya. Khinchin, Continued Fractions, Chapter 1, University of Chicago Press, Chicago, 1964.
6. E. Parzen, Modern Probability Theory and Its Applications, Wiley, New York, 1960.
7. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell, New York, 1961.

INTEGER POLYNOMIALS WITH PRESCRIBED INTEGER VALUES

CHARLES SMALL, Queen's University, Kingston

For a commutative ring R (with 1) we let $U(R)$ denote the multiplicative group of invertible elements of R. Define $S(R)$ to be the set of all α in R such that $\alpha + 2x$ is in $U(R)$ for some x in R, and $T(R)$ to be the set of all α in R such that $\alpha^2 + 4y$ is in $U(R)$ for some y in R. Clearly $S(R) \subseteq T(R)$: if $\alpha + 2x$ is invertible then so is $(\alpha + 2x)^2 = \alpha^2 + 4(\alpha + x)x$.

The subsets $S(R)$ and $T(R)$ arise in the study of Galois extensions of R; the rings R for which $S(R) \neq T(R)$ are precisely those which have free quadratic extensions which admit no normal basis (see [1]). Hence it is of interest to find rings R with $S(R) \neq T(R)$. A naive search quickly produces many rings for which $S(R) = T(R)$. The reader will have little trouble in seeing that this is the case, for example, when R is a field, or more generally a local ring; also when R is $\mathbb{Z}$ (ring of integers) or a polynomial ring in any number of variables over $\mathbb{Z}$, or $\mathbb{Z}/n\mathbb{Z}$ for any n; also whenever $2 \in U(R)$, and (at the other extreme) whenever $2 = 0$ in R (or more generally whenever 2 is in every maximal ideal of R).

Nonetheless there are plenty of rings R for which $S(R) \neq T(R)$. A colleague, N. Pullman, found the following example. Let R be the subring of $\mathbb{Z} \times \mathbb{Z}$ generated show that q can be considered as drawn uniformly from the set $\{1, 2, \cdots, p-1\}$. Given p, we have

$$P(q/p < c) = P(q < cp)$$
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where the square brackets denote the integral part. For large p, this fraction approaches c, so for large n, the probability of an arbitrarily chosen starting position being unsafe approaches c. That is, the first to move, player A, is the more likely to have a winning strategy available.

References

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Nonetheless there are plenty of rings R for which $S(R) \neq T(R)$. A colleague, N. Pullman, found the following example. Let R be the subring of $\mathbb{Z} \times \mathbb{Z}$ generated by (1,7), that is, the image of the homomorphism $\mathbb{Z}[X] \to \mathbb{Z} \times \mathbb{Z}$ defined by evaluation at (1,7). We claim that (1,7) itself is in $T(R)$ but not in $S(R)$. The first assertion is easy: it suffices to find an integer polynomial $g(X)$ with $1 + 4g(1) = 1 = 49 + 4g(7)$, and $g(X) = -(X-1)(X-5)$ is one such. ($g(X) = -2(X-1)$ is another.) To see that (1,7) is not in $S(R)$ we must show that no integer polynomial $f(X)$ can satisfy $1 + 2f(1) = \pm 1$ and $7 + 2f(7) = \pm 1$. If $f(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$, we have $f(7) - f(1) = 6a_1 + (49-1)a_2 + \cdots + (7^n - 1)a_n$. By induction, this is divisible by 6, because $7^i - 1 = 7(7^{i-1} - 1) + 6$. But with $1 + 2f(1) = \pm 1$ (i.e., $f(1) = 0$ or -1) and $7 + 2f(7) = \pm 1$ (i.e., $f(7) = -3$ or -4) we can never have $f(7) - f(1)$ divisible by 6.

This argument does more than show that Pullman's example works: it shows that for any integer polynomial $f(X)$, $f(7) - f(1)$ is divisible by $6 = 7 - 1$, and it suggests:

THEOREM 1. *Let $f(X)$ be any integer polynomial and let a and b be any integers. Then $a - b$ divides $f(a) - f(b)$.*

To prove this using the argument indicated one needs only the lemma that $a - b$ divides $a^i - b^i$ for all i, which follows by induction from the identity

$$2(a^i - b^i) = (a+b)(a^{i-1} - b^{i-1}) + (a-b)(a^{i-1} + b^{i-1}).$$

Having proved a theorem, our next step must be to ask "What about the converse?" and we find:

THEOREM 2. *If a, b, c, d are integers such that $a - b$ divides $c - d$, there is an integer polynomial $f(X)$ such that $f(a) = c$ and $f(b) = d$; in fact there is such an $f(X)$ of degree $\leqq 1$.*

To prove this, simply put $f(X) = d + t(X - b)$ where t is the integer $(c-d)/(a-b)$. (Looked at geometrically, $f(X)$ "is" the line through the points with coordinates (a,c) and (b,d); the coefficients are integers because the slope is t. This takes the mystery out of the fact, contained in Theorems 1 and 2, that if there is *any* integer polynomial $f(X)$ such that $f(a) = c$ and $f(b) = d$, then there is one of degree $\leqq 1$. We will see below that when there is an integer polynomial taking prescribed integer values at n prescribed integers, there is one of degree $\leqq (n-1)$.)

We now have a new description of the ring $R = \{f(1,7) \mid f \in \mathbb{Z}[X]\}$ considered above, namely $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \bmod 6\}$. This suggests how to manufacture a whole family of examples: for any positive integer n let $R_n = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \bmod n\}$. Then, provided $n \equiv 2 \bmod 4$ and $n > 2$, the element $(1, n+1)$ is in $T(R_n)$ but not in $S(R_n)$.

Theorem 2 begs to be generalized. Suppose we have distinct integers $a_1, a_2, \cdots, a_n$ and integers $b_1, b_2, \cdots, b_n$, not all equal. When is there an integer polynomial $f(X)$ such that $f(a_i) = b_i$ for all $i = 1, 2, \cdots, n$? Theorem 1 gives an obvious necessary condition: $a_i - a_j$ must divide $b_i - b_j$ for all $i \neq j$. Theorem 2 shows this is also sufficient, *if* $n = 2$. Is it still sufficient when $n > 2$? We will show that the answer is "no".

Recall first that if we allow *rational* coefficients, there is a *unique* polynomial $\phi(X)$ of degree $< n$ with $\phi(a_i) = b_i$ for all $i = 1, 2, \cdots, n$, namely

$$\phi(X) = \sum_{i=1}^{n} b_i \left(\prod_{i \neq j} \frac{X - a_j}{a_i - a_j} \right).$$

(This is the "Lagrange interpolation formula", and it is discussed in [**2**, Section 5.3] (Section 29 in the seventh edition of the original (German) version).) The coefficients of $\phi(X)$ can be computed (same reference), and we leave to the reader the exercise of showing that our divisibility condition ($a_i - a_j$ divides $b_i - b_j$ for all $i \neq j$) is *not* sufficient to make them all integers. (It is enough, of course, to find a counterexample, say with $n = 3$.)

It is conceivable, however, that we might still achieve $f(a_i) = b_i$ for all i, where $f(X)$ has *integer* coefficients, by considering polynomials $f(X)$ of degree $\geqq n$. We will show that this freedom is actually illusory: if there is an integer polynomial $f(X)$ with $f(a_i) = b_i$ for all $i = 1, 2, \cdots, n$ then there is such an $f(X)$ of degree $< n$. (This means that if *any* integer polynomial works, the coefficients of the Lagrange polynomial $\phi(X)$ are integers.)

The trick to showing that passing to higher degrees is of no avail lies in avoiding the "natural" basis $1, X, X^2, \cdots$ for the polynomials, and sticking instead to the kind of basis suggested by Lagrange's formula. For example the polynomials

$$\begin{aligned} Y_0(X) &= 1,\ Y_1(X) = (X - a_1), \\ Y_2(X) &= (X - a_1)(X - a_2), \cdots, Y_n(X) = \prod_{i=1}^{n} (X - a_i), \\ Y_{n+1}(X) &= Y_n(X)X, \cdots, Y_{n+j}(X) = Y_n(X)X^j, \cdots, \end{aligned}$$

form a perfectly good basis, and clearly $\tilde{f}(X) = \sum_{0 \leqq i < n} c_i Y_i(X)$ takes exactly the same values at $a_1, a_2, \cdots, a_n$ as any given polynomial $f(X) = \sum_{0 \leqq i} c_i Y_i(X)$, for the deleted basis elements $Y_i(X)$, $i \geqq n$, vanish at every a_i.

Finally, let us return to the question of finding rings R for which $S(R) \neq T(R)$. The family of examples R_n, $n = 6, 10, 14, \cdots$, led above to some amusing computations with integer polynomials, but these rings seem somewhat artificial in the context in which the problem arose (Galois extensions of rings). A family of examples of much greater interest from this point of view can be constructed as follows. Let m be a square-free integer and let R_m be the ring of integers in the quadratic number field $\mathbb{Q}(\sqrt{m})$ ($\mathbb{Q}$ = rationals). Then, whenever $m < -1$ and $m \equiv 3 \bmod 4$, $2 + \sqrt{m}$ is in $T(R_m)$ but not in $S(R_m)$. (To check the second assertion one needs to know that when $-1 > m \equiv 3 \bmod 4$, $U(R_m) = \{\pm 1\}$.)

References

1. C. Small, Normal bases for quadratic extensions, to appear in the Pacific Journal of Mathematics.

2. B. van der Waerden, Algebra, vol. I, Ungar, New York, 1970.

A NOTE ON CHEBYSHEV'S THEOREM

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Chebyshev's theorem states that there are positive constants A and B such that $Ax < \Psi(x) < Bx$, $x \geqq 2$, where $\Psi(x) = \Sigma_{n \leqq x}\Lambda(n)$, and

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^{\alpha}, \text{ prime } p, \alpha \geqq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Shapiro [3] has shown that the existence of the constant B implies that of A, so that Chebyshev's theorem is a consequence of the result

$$\Psi(x) = O(x). \tag{1}$$

Although several proofs of (1) are known, many of those proofs are quite lengthy. While it is possible to gain elegance by first developing a convolution theory, it is undesirable in an introductory course in number theory to spend the time required to develop this heavy machinery. In this note, we give a proof of (1) which, though certainly not the most elegant, is shorter than many. Moreover, this proof has the pedagogic advantage of using a single theorem (stated below) to derive the principal lemmas needed (see (4) and (5)), as well as to prove (1) itself.

THEOREM 1. *Suppose μ is the Möbius function, and f and g are functions of a real variable. Then*

$$f(x) = \sum_{n \leqq x} g(x/n) \Leftrightarrow g(x) = \sum_{n \leqq x} \mu(n) f(x/n).$$

(For a proof, see [**1**; p. 79] or [**2**; p. 237]. Also, Spears [**4**] has proved the implication from left to right, which is all that is required in what follows.)

We use these known results:

$$\sum_{n \leqq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right), \quad \gamma \text{ is Euler's constant,} \tag{2}$$

$$\sum_{n \leqq x} \log n = x \log x - x + O(\log x). \tag{3}$$

Both of these formulas are easy deductions from the Euler-Maclaurin sum formula. Also, we need the equation $\log n = \Sigma_{d|n}\Lambda(d)$.

Taking $g(x) = 1$ in Theorem 1, we get

$$M_1(x) \equiv \sum_{n \leqq x} \frac{\mu(n)}{n} = O(1). \tag{4}$$

Taking $g(x) = x$ in Theorem 1, with (2) we get

$$M_2(x) \equiv \sum_{n \leqq x} \frac{\mu(n)}{n} \log \frac{x}{n} = O(1). \tag{5}$$

Since

$$\sum_{n\leqq x}\log n=\sum_{n\leqq x}\sum_{d|n}\Lambda(d)=\sum_{n\leqq x}\sum_{d\leqq x/n}\Lambda(d)=\sum_{n\leqq x}\Psi\left(\frac{x}{n}\right),$$

by comparing this with (3), we have

$$\sum_{n\leqq x}\Psi\left(\frac{x}{n}\right)=x\log x-x+O(\log x).$$

Finally, we use Theorem 1 again, with $g(x)=\Psi(x)$ and $f(x)=x\log x-x+O(\log x)$, to conclude that

$$\begin{aligned}\Psi(x)&=\sum_{n\leqq x}\mu(n)\left\{\frac{x}{n}\log\frac{x}{n}-\frac{x}{n}+O\left(\log\frac{x}{n}\right)\right\}\\&=xM_2(x)-xM_1(x)+O\left(\sum_{n\leqq x}\log\frac{x}{n}\right)\\&=O(x)+O(x)+O\left(\sum_{n\leqq x}\log x-\sum_{n\leqq x}\log n\right)\\&=O(x).\end{aligned}$$

References

1. A. A. Gioia, The Theory of Numbers: An Introduction, Markham, Chicago, 1970.
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., Oxford University Press, London, 1962.
3. H. N. Shapiro, On the number of primes less than or equal to x, Proc. Amer. Math. Soc., 1 (1950) 346–348.
4. Nina Spears, A simplified proof of Chebyshev's theorem, Duke Math. J., 37 (1970) 709–713.

ON PALINDROMES

HEIKO HARBORTH, TU Braunschweig, Germany

In g-adic representation ($g\geqq 2$) we consider a positive integer

$$n_0=\sum_{i=0}^{m}d_ig^i=d_md_{m-1}\cdots d_1d_0,\qquad 0\leqq d_i\leqq g-1,$$

and the corresponding mirror number $\tilde{n}_0$ (with the digits of n_0 written in reverse order). The sum of these two numbers gives a new one $n_1=n_0+\tilde{n}_0$. This procedure leads to an infinite sequence $\{n_k\}$ of increasing integers $n_k=n_{k-1}+\tilde{n}_{k-1}$. Now it is a conjecture [**1**], that every such sequence produced by an arbitrary n_0 has at least one element as a palindrome ($n=\tilde{n}$). C.W. Trigg [**3**] checked all integers $<10{,}000$ in the decimal notation ($g=10$). For 249 of them he found neither a palindrome nor any regularity showing the conjecture to be false. In this paper we will construct infinitely many sequences which are counterexamples in bases $g=2^\alpha$ with $\alpha=1,2,3,\cdots$. In the case $\alpha=1$, a special sequence of this kind ($n_0=22$) already is given in [**2,4,5**], and two further ($n_0=77$ and 775) were found in [**5**].

Since

$$\sum_{n\leqq x}\log n=\sum_{n\leqq x}\sum_{d|n}\Lambda(d)=\sum_{n\leqq x}\sum_{d\leqq x/n}\Lambda(d)=\sum_{n\leqq x}\Psi\left(\frac{x}{n}\right),$$

by comparing this with (3), we have

$$\sum_{n\leqq x}\Psi\left(\frac{x}{n}\right)=x\log x-x+O(\log x).$$

Finally, we use Theorem 1 again, with $g(x)=\Psi(x)$ and $f(x)=x\log x-x+O(\log x)$, to conclude that

$$\begin{aligned}\Psi(x) &= \sum_{n\leqq x}\mu(n)\left\{\frac{x}{n}\log\frac{x}{n}-\frac{x}{n}+O\left(\log\frac{x}{n}\right)\right\}\\ &= xM_2(x)-xM_1(x)+O\left(\sum_{n\leqq x}\log\frac{x}{n}\right)\\ &= O(x)+O(x)+O\left(\sum_{n\leqq x}\log x-\sum_{n\leqq x}\log n\right)\\ &= O(x).\end{aligned}$$

References

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Letting $(d)_i$ denote the digit d repeated i times, for any positive integer y we start with

$$n_0 = 1\,0(g-1)_y\, g-1\; g-2\; g-1\,(0)_y 0\,0.$$

Then for $1 \leqq j \leqq \alpha - 1$ we get

$$n_j = 2^{j-1}\,2^{j-1}\,(0)_y\, g-2^{j-1}\; g-2^j-1\; g-2^{j-1}-1\,(g-1)_y 2^{j-1}-1\,2^{j-1},$$

and

$$n_\alpha = 2^{\alpha-1}\,2^{\alpha-1}\,(0)_y 2^{\alpha-1}-1\; g-1\; 2^{\alpha-1}-1\,(g-1)_y 2^{\alpha-1}-1\,2^{\alpha-1}.$$

Starting with $n_{\alpha+1}$ we have one more digit.

$$n_{\alpha+1} = 1\,0(g-1)_y\, g-1\; g-1\; g-2\; g-1\;(0)_y 0\,0,$$

$$n_{\alpha+2+j} = 2^j\,2^j\,(0)_y\; g-2^j\; g-2^j-1\; g-2^j-1\; g-2^j-1\;(g-1)_y\; 2^j-1\; 2^j$$

for $0 \leqq j \leqq \alpha - 1$. Finally, $n_{2\alpha+2}$ has two more digits than n_0 and is of the same form as n_0, with $y+1$ repetitions instead of y for the digits $g-1$ and 0.

$$n_{2\alpha+2} = 1\,0(g-1)_{y+1}\; g-1\; g-2\; g-1\;(0)_{y+1} 0\,0.$$

So in a certain sense there is a periodicity of length $2\alpha+2$, that is, $n_{(2\alpha+2)r+j}$ will become like n_j with $y+r$ instead of y $(0 \leqq j \leqq 2\alpha+1)$. Furthermore two numbers of one period (n_0 and $n_{\alpha+1}$) begin with 1 and end with 0, and the remaining n_j differ in the second positions from both sides. So a palindrome never occurs, and we have found one sequence contradicting the conjecture. Figure 1 shows, for example, one period in the case $g = 4$. The triples of points everywhere mean the same number $(y-2)$ of those digits being to the right and to the left of the points.

```
   1 0 3 ··· 3 3 2 3 0 ··· 0 0 0
   0 0 0 ··· 0 3 2 3 3 ··· 3 0 1
   -----------------------------
   1 1 0 ··· 0 3 1 2 3 ··· 3 0 1
   1 0 3 ··· 3 2 1 3 0 ··· 0 1 1
   -----------------------------
   2 2 0 ··· 0 1 3 1 3 ··· 3 1 2
   2 1 3 ··· 3 1 3 1 0 ··· 0 2 2
  -------------------------------
 1 0 3 ··· 3 3 3 2 3 0 ··· 0 0 0
 0 0 0 ··· 0 3 2 3 3 3 ··· 3 0 1
 -------------------------------
 1 1 0 ··· 0 3 2 2 2 3 ··· 3 0 1
 1 0 3 ··· 3 2 2 2 3 0 ··· 0 1 1
 -------------------------------
 2 2 0 ··· 0 2 1 1 1 3 ··· 3 1 2
 2 1 3 ··· 3 1 1 1 2 0 ··· 0 2 2
---------------------------------
1 0 3 3 ··· 3 3 2 3 0 ··· 0 0 0 0
```

FIG. 1.

For every integer $x \geqq 0$ we now take into account the sequences $\{n_k(x)\}$ being produced by

$$n_0(x) = 1\,0\,(g-1)_y\; g-1\; g-2\; g-1\; (0)_x\,0\,0\,1\,0\,(g-1)_x\; g-1\; g-2\; g-1\; (0)_y\,0\,0.$$

Similar arguments as above lead us to the fact that these sequences are also periodic with length $2\alpha + 2$, and that no element becomes palindromic. The two blocks with x digits 0 and with x digits $g - 1$ remain constant for every element of one sequence, whereas each of the two blocks with y digits, period by period, increases by one digit of the same type. As every x gives another sequence, we have the following theorem, where two sequences are called different, if every element of one sequence is different from every element of the other one.

THEOREM. *In every g-adic number system with* $g = 2^\alpha$, $\alpha = 1,2,3,\cdots$, *there are infinitely many pairwise different sequences* $\{n_k\}$ *without any palindromes.*

For example, Figure 2 shows one period of these sequences in the case $\alpha = 3$. The pairs, respectively triples, of points mean $x - 2$, respectively $y - 2$, of those digits being to the left and to the right of the points.

There are other similar types of sequences which are periodic and without palindromes in bases $g = 2^\alpha$. Therefore the question arises, whether in these number systems every sequence $\{n_k\}$ produced by any n_0 after finite steps of iteration will be periodic in the above sense and without palindromes.

```
     1 0 7 ··· 7 7 6 7 0 ·· 0 0 0 1 0 7 ·· 7 7 6 7 0 ··· 0 0 0
     0 0 0 ··· 0 7 6 7 7 ·· 7 0 1 0 0 0 ·· 0 7 6 7 7 ··· 7 0 1
     ---------------------------------------------------------
     1 1 0 ··· 0 7 5 6 7 ·· 7 0 1 1 1 0 ·· 0 7 5 6 7 ··· 7 0 1
     1 0 7 ··· 7 6 5 7 0 ·· 0 1 1 1 0 7 ·· 7 6 5 7 0 ··· 0 1 1
     ---------------------------------------------------------
     2 2 0 ··· 0 6 3 5 7 ·· 7 1 2 2 2 0 ·· 0 6 3 5 7 ··· 7 1 2
     2 1 7 ··· 7 5 3 6 0 ·· 0 2 2 2 1 7 ·· 7 5 3 6 0 ··· 0 2 2
     ---------------------------------------------------------
     4 4 0 ··· 0 3 7 3 7 ·· 7 3 4 4 4 0 ·· 0 3 7 3 7 ··· 7 3 4
     4 3 7 ··· 7 3 7 3 0 ·· 0 4 4 4 3 7 ·· 7 3 7 3 0 ··· 0 4 4
   -----------------------------------------------------------
   1 0 7 ··· 7 7 7 6 7 0 ·· 0 0 1 0 7 ·· 7 7 7 6 7 0 ··· 0 0 0
   0 0 0 ··· 0 7 6 7 7 7 ·· 7 0 1 0 0 ·· 0 7 6 7 7 7 ··· 7 0 1
   -----------------------------------------------------------
   1 1 0 ··· 0 7 6 6 6 7 ·· 7 0 2 1 0 ·· 0 7 6 6 6 7 ··· 7 0 1
   1 0 7 ··· 7 6 6 6 7 0 ·· 0 1 2 0 7 ·· 7 6 6 6 7 0 ··· 0 1 1
   -----------------------------------------------------------
   2 2 0 ··· 0 6 5 5 5 7 ·· 7 1 4 2 0 ·· 0 6 5 5 5 7 ··· 7 1 2
   2 1 7 ··· 7 5 5 5 6 0 ·· 0 2 4 1 7 ·· 7 5 5 5 6 0 ··· 0 2 2
   -----------------------------------------------------------
   4 4 0 ··· 0 4 3 3 3 7 ·· 7 4 0 4 0 ·· 0 4 3 3 3 7 ··· 7 3 4
   4 3 7 ··· 7 3 3 3 4 0 ·· 0 4 0 4 7 ·· 7 3 3 3 4 0 ··· 0 4 4
   -----------------------------------------------------------
 1 0 7 7 ··· 7 7 6 7 0 ·· 0 0 0 1 0 7 ·· 7 7 6 7 0 ··· 0 0 0 0
```

FIG. 2.

References

1. D. Lehmer, Sujets d'étude, Sphinx (Bruxelles), 8 (1938) 12–13.
2. R. Sprague, Unterhaltsame Mathemathik, Vieweg, Braunschweig, 1965, pp. 5–6, pp. 25–26.
3. C. W. Trigg, Palindromes by addition, this MAGAZINE, 40 (1967) 26–28.
4. H. Gabai and D. Coogan, On palindromes and palindromic primes, this MAGAZINE, 42 (1969) 252–254.
5. Brother A. Brousseau, Palindromes by addition in base two, this MAGAZINE, 42 (1969) 254–256.

CONSTRUCTING A THIRD ORDER MAGIC SQUARE

CHARLES W. TRIGG, San Diego, California

In a third order magic square the triads in the rows, columns, and unbroken diagonals have the same magic sum, which is three times the central element.

It is well known, see the reference for example, that the members of three three-term arithmetic progressions, with the same common difference and with their leading terms in arithmetic progression, can be arranged to form a magic square. This arrangement can be accomplished in two easy operations, starting with the progressions in a square array,

$$\begin{array}{lll} x & x+a & x+2a \\ x+b & x+a+b & x+2a+b. \\ x+2b & x+a+2b & x+2a+2b \end{array}$$

Since each triad in a straight line which contains the central element is in arithmetic progression and has the magic sum, $3(x+a+b)$, then

1. Interchange the elements at the extremities of each diagonal, or those at the extremities of each bimedian;
2. Rotate (clockwise or counterclockwise) the perimeter one space.

Either of these operations may be performed first. Different orientations of the same magic square may be obtained, depending upon the choice of the operations. For example:

$$\begin{array}{ccc} 1 & 3 & 5 \\ 4 & 6 & 8 \\ 7 & 9 & 11 \end{array} \rightarrow \begin{array}{ccc} 4 & 1 & 3 \\ 7 & 6 & 5 \\ 9 & 11 & 8 \end{array} \rightarrow \begin{array}{ccc} 8 & 1 & 9 \\ 7 & 6 & 5 \\ 3 & 11 & 4 \end{array} \text{ or } \begin{array}{ccc} 4 & 11 & 3 \\ 5 & 6 & 7. \\ 9 & 1 & 8 \end{array}$$

The twist '*n*' trade or trade '*n*' twist routine applied to any third order magic square, in turn produces a square array of arithmetic progressions.

Reference

Maurice Kraitchik, Mathematical Recreations, Dover, New York, 1953, p. 148.

References

1. D. Lehmer, Sujets d'étude, Sphinx (Bruxelles), 8 (1938) 12–13.
2. R. Sprague, Unterhaltsame Mathemathik, Vieweg, Braunschweig, 1965, pp. 5–6, pp. 25–26.
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NORM PRESERVING OPERATORS ON DECOMPOSABLE TENSORS

RICHARD BRONSON, Fairleigh Dickinson University

1. Introduction. Whenever possible, one characterizes global properties of a vector space or a linear transformation by specifying requirements only on a subset of the vector space, most generally a basis. In this regard, the following open question was presented at the 1971 NSF Summer Institute on Algebra at the University of California at Santa Barbara.

PROPOSITION. *Let V be an n-dimensional inner product space over the complex numbers and let L be a linear transformation from the p-fold tensor product $V \otimes V \otimes \cdots \otimes V$ into itself. If*

$$\| L(v_1 \otimes v_2 \otimes \cdots \otimes v_p) \| = \| v_1 \otimes v_2 \otimes \cdots \otimes v_p \|$$

for every decomposable element in $V \otimes V \otimes \cdots \otimes V$, then L is unitary.

The proposition is false. Since $V \otimes V$ is isomorphic to $M_n(C)$, the set of all $n \times n$ matrices with complex coefficients, and each decomposable element in $V \otimes V$ can be represented by a matrix of rank one, it suffices to contradict the following conjecture: Let $L: M_n(C) \to M_n(C)$ be a linear transformation. If $\| L(A) \| = \| A \|$ for every $A \in M_n(C)$ having rank one, then L is unitary.

2. A counterexample. Take $n = 2$. Every 2×2 matrix is isomorphic to a 4-tuple, hence L is isomorphic to a 4×4 matrix. Consider

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Using the standard Euclidean norm and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b, c, d)^T,$$

where superscript T denotes transposition, we calculate that

$$\| L(A) \| = [a^2 + b^2 + c^2 + d^2 + 2(ad - bc)]^{\frac{1}{2}}.$$

Thus, for any A having rank one, $ad - bc = 0$, and $\| L(A) \| = \| A \|$. But since the first and fourth columns of L are not orthogonal, L is not unitary.

Acknowledgement. The author is indebted to the referee for simplifications in the counterexample.

THE CONNECTION OF BLOCK DESIGNS WITH FINITE BOLYAI-LOBACHEVSKY PLANES

G. SPOAR, University of Wyoming

In [2] S. H. Heath studied systems called finite Bolyai-Lobachevsky planes (hyperbolic planes) for which given any point P not on a line l, there are exactly m ($m \geqq 2$) lines through P which are parallel to l (two lines which do not have a point in common will be called parallel).

We recall that if there exists one line containing exactly k points, then we have the following:

(a) Every line contains exactly k points.

(b) There are exactly $r = m + k$ lines which pass through each point (the significance of r will become clear later).

(c) There are exactly $r(k-1)+1$ points.

(d) There are exactly $[r(k-1)+1][r/k]$ lines.

(e) If $(r-k)(r-k-1)$ is not divisible by k, there is no finite $BL(r-k,k)$ geometry (the symbol $BL(r-k,k)$ will denote the finite Bolyai-Lobachevsky plane with k points on each line and $r-k$ parallels to a line through each point not on that line).

In his paper Heath determined the existence of several infinite classes of such systems. During his treatment some natural questions were raised:

1. Do finite $BL(r-k,k)$ exist for $k > r-k$? In particular does the smallest possible Bolyai-Lobachevsky plane ($k > r-k$), namely $BL(3,6)$, exist?

2. Does $BL(r-3,3)$ exist for each integer r not ruled out by (e) above?

3. Are there any values of r and k for which $BL(r-k,r)$ do not exist excepting those ruled out by (e)?

We will now show that another known approach is very useful in analyzing existence questions of this nature. This is the theory of (b,v,r,k,λ) designs [1].

DEFINITION. *A (balanced incomplete block) design is an arrangement of v distinct objects into b blocks such that each block contains exactly k distinct objects, each object occurs in exactly r different blocks, and every pair of distinct objects a_i, a_j occurs together in exactly λ blocks.*

It is easily seen that there are two necessary relations on the five parameters:

$$\begin{aligned} bk &= vr, \\ r(k-1) &= \lambda(v-1). \end{aligned} \tag{1}$$

A design is described by its incidence matrix $A = (a_{ij})$, $i = 1,2,\cdots,v$, $j = 1,2,\cdots,b$ determined as follows. If $a_1, a_2, \cdots, a_v$ are the objects and $B_1, B_2, \cdots, B_b$ are the blocks then

$$a_{ij} = 1 \quad \text{if} \quad a_i \in B_j,$$

$$a_{ij} = 0 \quad \text{if} \quad a_i \notin B_j.$$

Now if we consider the objects as points and the blocks as lines then it is easy to conclude that the existence of $BL(r-k,k)$ planes is equivalent to the existence of $(b,v,r,k,1)$ designs. In particular $v = r(k-1)+1$ and $b = [r(k-1)+1][r/k]$ are obtained directly from the necessary conditions (1) with $\lambda = 1$. Moreover it can be readily verified that the divisibility criterion in (e) follows from (1) with $\lambda = 1$.

In regard to this latter approach the question of the existence of designs has been studied extensively. With regard to the questions formulated by Heath earlier,

1. $(b,v,r,k,1)$ designs with $k > r-k$ remain unknown. In particular the design $(69,46,9,6,1)$ is unknown and appears difficult to determine.

2. $(b,v,r,3,1)$ designs exist for each integer r not ruled out by the divisibility criterion in (e). Such systems are called Steiner triples [**1**, 15.4]. Moreover, $(b,v,r,4,1)$ designs exist for all possible r not ruled out by (e) [**1**, 15.5].

3. In [**2**] the existence of all possible $BL(r-3,3)$ and $BL(r-4,4)$ planes was established. For $k \geqq 5$ the existence problem of $(b,v,r,k,1)$ designs generally remains unsolved. In particular, one could ask the following question.

"Do all $(b,v,r,5,1)$ designs exist (where r is not ruled out by (e)) or, equivalently, do all $BL(r-5,5)$ planes exist?"

In regard to this last question it is known that all exist except for finitely many [**3**].

The author is currently at the University of Guelph, Guelph, Ontario, Canada.

References

1. M. Hall Jr., Combinatorial Theory, Blaisdell, Waltham, 1967.

2. S. H. Heath, The existence of finite Bolyai-Lobachevsky planes, this MAGAZINE, 43 (1970) 244-249.

3. R. M. Wilson, An existence theory for pairwise balanced designs, Ph. D. dissertation, Department of Mathematics, Ohio State University, 1969.

CEM AND THE MAA FILM PROJECTS ADVISORY COMMITTEE

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If you can contribute any first-hand information on the film rental service but have not received a questionnaire, please send such data — favorable or unfavorable — to Prof. J. D. E. Konhauser, Department of Mathematics, Macalester College, St. Paul, Minnesota 55101.

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PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before September 1, 1973.

PROPOSALS

859. *Proposed by B. Suer and H. Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$THREE + NINE = EIGHT + FOUR.$$

860. *Proposed by Leon Bankoff, Los Angeles, California.*

In any triangle ABC show that

$$\sin A/2 + \sin B/2 + \sin C/2 \geqq \cos A + \cos B + \cos C.$$

861. *Proposed by Erwin Just, Bronx Community College.*

Find all integral values of m for which the function, f, defined by $f(x) = x^3 - mx^2 + mx - (m^2 + 1)$ has an integral zero.

862. *Proposed by K. R. S. Sastry, Makele, Ethiopia.*

It is well known that the bisectors of equal angles of an isosceles triangle are equal. Conversely, the Steiner-Lehmus theorem states that if two internal bisectors of a triangle are equal, the triangle is isosceles. Show that if two external bisectors of a triangle are equal, the triangle need not be isosceles.

863. *Proposed by K. W. Schmidt, University of Manitoba, Canada.*

The number of (-1)'s of an n-order Hadamard matrix is bounded by $n[n \pm (\sqrt{2n-1} - 1)]/2$.

864. *Proposed by Charles W. Trigg, San Diego, California.*

In the square array

1	2	3
4	5	8
7	6	9

all but two of the twelve adjacent digit pairs, taken horizontally and vertically, have prime sums.

(a) Show that it is impossible to rearrange the digits so that every pair of adjacent digits has a prime sum.

(b) Show that the digits can be rearranged so that each of the twelve sums of adjacent digits is composite, with nine of the composite sums being distinct.

865. *Proposed by A. Polter Geist, Miskatonic University, Arkham, Massachusetts.*

A square tract of land is bounded by four roads. A barn is located on the tract exactly 13 miles from the southwest corner, 8 miles from the northwest corner, and 5 miles from the northeast corner. How far is the barn from the nearest road?

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q562. Let $f(z)$ be a bounded entire function. Then $f(z)$ is constant.

[*Submitted by Warren Page*]

Q563. Let H be a nontrivial subgroup of the multiplicative group of integers $G = \{1, 2, \cdots p-1\}$ modulo p where p is a prime. Prove that the sum of the elements of H is divisible by p.

[*Submitted by Erwin Just and Helen Jick*]

Q564. If $A_iB_iC_iD_i$ $(i = 1, 2, 3, 4)$ denote four given quadrilaterals in space such that the four vector sums

$$A_iB_{i+1} + C_iB_{i-1} + A_{i+1}B_{i+2} + C_{i+1}D_{i+2} + A_{i+2}B_i + C_{i+2}D_i,$$

$$(i = 1, 2, 3, 4 \text{ and } A_i = A_{i+4}, \text{ etc.})$$

are zero, show that the sums remain zero for any changes of the orientation of the quadrilaterals.

[*Submitted by Murray S. Klamkin*]

Q565. Determine the trihedral angles $O\text{-}A'B'C'$ such that if one picks an arbitrary point A, B, C, respectively, on the open rays OA', OB', OC', then ABC is always an acute triangle.

[*Submitted by Murray S. Klamkin*]

Q566. The arithmetic mean of twin primes 5 and 7 is a perfect number 6. Are there other twin primes with a perfect mean?

[*Submitted by Charles W. Trigg*]

(Answers on page 112)

SOLUTIONS

Late Solutions

Jorge Andres, St. Francis College, New York: **827**; *Edward D. Bender, Bradley University, Illinois:* **827**; *Benjamin Bock, Trinity College, Connecticut:* **824**; *Berne Switzerland Problem Solving Group:* **829**; *William F. Dwyer, Queens College, New York:* **826, 827**; *Joe Flower, Northeast Missouri State University:* **824, 829**; *M. G. Greening, University of New South Wales, Australia:* **826, 827, 828, 830**; *Ray Haertel, Central Oregon Community College:* **827**; *Shiv Kumar and Miss Nirmal (jointly), Ohio State University:* **827, 829**; *Mrs. Jimmse Mandel, Central Oregon Community College:* **824**; *Joseph V. Michalowicz, Catholic University of America:* **826**; *Albert J. Patsche, Rock Island Arsenal, Illinois:* **827**; *Thomas J. Pennello, Michael J. Fay, Jeffrey T. Moore (jointly), University of Santa Clara, California:* **824**; *Bob Prielipp and N. J. Kuenzi (jointly), University of Wisconsin-Oshkosh:* **830**; *C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India:* **827.**

A Unique Solution

831. [May, 1972] *Proposed by Gary R. Gruber, Hofstra University, New York.*

Consider the functional equation holding for all x, y such that $F(x, y)$ is bounded and $f(x + y) - f(x) = y^2F(x, y)$. Find an explicit solution (that is, find the functions $f(x)$ and $F(x, y)$, excepting $F(x, 0)$ which is indeterminate) and show that the solution is unique.

Solution by Lester Rubenfeld, Michael Frame, David Hale, Clayton Keller, David Rutledge, Stephen Sentoff and David Wagner, Rensselaer Polytechnic Institute (National Science Foundation Undergraduate Research Participants during the summer of 1972).

Given: $f(x + y) - f(x) - y^2F(x, y)$.
Assuming $F(x, y)$ is bounded, we have

$$\frac{f(x+y)-f(x)}{y} = y\ F(x,y)$$

$$\lim_{y\to 0}\frac{f(x+y)-f(x)}{y} = \lim_{y\to 0} yF(x,y) = 0$$

$$\therefore\ f'(x) \equiv 0,$$

$$\therefore\ f(x) = C.$$

Substitution of f in the original equation gives

$$y^2F(x, y) = C - C = 0, \qquad \text{for all } x, y.$$

Hence

$$F(x, y) \equiv 0 \quad \text{where} \quad y \neq 0.$$

The solution is unique insofar as antidifferentiation yields an arbitrary constant.

Also solved by Lowell W. Beineke, Purdue University at Fort Wayne, Indiana; Gary L. Britton, University of Wisconsin at West Bend; Ralph Garfield, The College of Insurance, New York; Barbara A. Keller, Fort Wayne, Indiana; Alexander Kleiner, Drake University; Shiv Kumar and Miss Nirmal (jointly), Ohio State University; Stephen K. Park, Langley Research Center, Hampton, Virginia; Problem Solving Group, Berne, Switzerland; Phil Tracy, Liverpool, New York; Julius Vogel, Newark, New Jersey; and the proposer.

Casting Out Nines

832. [May, 1972] *Proposed by Guy A. R. Guillot, Montreal, Canada.*

A certain six-digit integer when divided by an even two-digit integer gives a result in which the sum of the 1st, 2nd, 3rd, and 4th product is equal to the first, second, third and fourth difference digits in the quotient, respectively. If there is no remainder, what is the prime quotient?

```
X X ) X X X X X X ( X X X
      X X X
      -----
        X X X
        X X X
        -----
          X X X
          X X X
          -----
            X X X
            X X X
            -----
```

Solution by Charles W. Trigg, San Diego, California.

Presumably "difference" should have been "different" and "digits of" was omitted before "1st".

For $k < 9$, the only even two-digit d's such that the digits of the three-digit kd's sum to k are: $k = 3$, $d = 34, 40, 70$; $k = 4$, $d = 28$; $k = 5$, $d = 28, 46, 64, 82$; $k = 6$, $d = 22, 34, 40, 52, 70$; $k = 7$, $d = 46$; and $k = 8$, $d = 28, 64$. The only d associated with three k's is 28, and since $9(28) = 252$, the quotient is a permutation of the digits 4, 5, 8 and 9. Their only prime permutation is 5849, the desired quotient.

The reconstructed division is

```
28) 163772 (5849
    140
     237
     224
      137
      112
       252
       252
```

Also solved by Kenneth M. Wilke, Topeka Kansas, who made "different basic assumptions," and the proposer.

Establishing an Inequality

833. [May, 1972] *Proposed by A. K. Gupta, University of Arizona.*

Let $a_n = (n!)^{1/n}$ where n is a positive integer. Consider the sequence $\{b_n\}$ where $b_n = a_{n+1}/a_n$. Show that $b_n > 2^{1/n+1}$ for all $n > 1$.

I. *Solution by M. T. Bird, California State University at San Jose.*

The arithmetic mean of n distinct positive numbers exceeds their geometric mean. Consequently for the numbers $1, 2 \cdots n$ with $n > 1$ we have

$$(n+1)/2 > (n!)^{1/n}.$$

Multiplying both members by $n!$ yields

$$(n+1)!/2 > (n!)^{(n+1)/n},$$

$$(n+1)! > 2(n!)^{(n+1)/n},$$

$${a_{n+1}}^{n+1} > 2\ a_n^{n+1}$$

$$b_n^{n+1} > 2$$

and finally for $n > 1$

$$b_n > 2^{1/(n+1)}.$$

II. *Solution by Julius Vogel, Prudential Insurance Company, Newark, New Jersey.*

The proof is by induction. We observe that

$$b_2 = (3!)^{1/3}/(2!)^{1/2} = 3^{1/3} \cdot 2^{-1/6} = (3 \cdot 2^{-1/2})^{1/3} > 2^{1/3}$$

Suppose

$$b_k > 2^{1/k+1}.$$

This inequality readily transforms into

(a) $$(k+1)^k > 2^k k!$$

We note next that

(b) $$(1 + 1/k+1)^{k+1} > 2.$$

Multiplying inequality (a) by $(k+1)$ and then by inequality (b) yields

$$(k+2)^{k+1} > 2^{k+1}(k+1)!$$

This in turn transforms back into

$$b_{k+1} > 2^{1/k+2}$$

completing the induction.

Also solved Mitrovski Cvetan, B. Miladinovci, Bitola, Yugoslavia; Ragnar Dybvik, Tingvoll, Norway; Leon Gerber, St. John's University; Michael Goldberg, Washinton, D. C.; M. G. Greening, University of New South Wales, Australia; Richard A. Groeneveld, Iowa State University; Elkedagmar Heinrich, Frankfurt, Germany; Steven Janke, University of California at Berkeley; Will Kazez, Pennsylvania State University; Charles Kimble, East St. Louis, Illinois; Brother Brendan Kneale, St. Mary's College, California; Vaclav Konecny, Jarvis Christian College; Lew Kowarski, Morgan State College, Maryland; Norbert J. Kuenzi, Oshkosh, Wisconsin; Maurice Mizrahi, University of Texas at Austin; Edward Moylan, Plymouth, Michigan; C.B.A. Peck, State College, Pennsylvania; Bob Prielipp, University of Wisconsin at Oshkosh; Lawrence A. Ringenberg, Eastern Illinois University; Barry H. Rodin, Aberdeen Proving Ground, Maryland; K.R.S. Sastry, Makele, Ethiopia; Phil Tracy, Liverpool, New York; Wolf R. Umbach, Rottorf, West Germany; C. S. Venkataraman, Sree Kerala Varma College, Trichur, India; Kenneth Wilke, Topeka, Kansas; and the proposer.

Divisors of a Set

834. [May, 1972] *Proposed by Marion B. Smith, University of Wisconsin, Baraboo, Wisconsin.*

Let c be a positive integer and define a set S_c as follows: $S_c = \{(x,y) \mid x$ and y are positive or negative integers and $1/x + 1/y = 1/c\}$. Prove that $\Sigma_{(x,y)\in S_c}(x+y) = 4c\tau(c^2)$ where $\tau(c^2)$ is the number of divisors of c^2.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh.

We begin by noting that $1/x + 1/y = 1/c$ is equivalent to $xy = cx + cy$. Next we establish that $1/x + 1/y = 1/c$ has $2\tau(c^2) - 1$ integer solutions. Let $x = c + a$ and $y = c + b$ where a and b are integers. Then $ab = (x-c)\,(y-c) = xy - cx - cy + c^2 = 0 + c^2 = c^2$. There are $\tau(c^2)$ positive values of a that satisfy $ab = c^2$. Also there are $\tau(c^2)$ negative values of a that satisfy $ab = c^2$, but one of these is $a = -c$ which makes $x = 0$. Thus $1/x + 1/y = 1/c$ has $2\tau(c^2) - 1$ integer solutions.

Therefore $\Sigma_{(x,y)\in S_c}\,(x+y) = \Sigma_{(x,y)\in S_c}\,(2c + (a+b)) = \Sigma_{(x,y)\in S_c}\,2c + \Sigma_{(x,y)\in S_c}\,(a+b) = (2\tau(c^2)-1)2c + \Sigma_{\substack{(x,y)\in S_c\\ a\neq b}}\,(a+b) + \Sigma_{\substack{(x,y)\in S_c\\ a=b}}\,(a+b) = [4c\tau(c^2) - 2c] + 0 + 2c = 4c\tau(c^2)$.

Also solved by Richard J. Bonneau; L. Carlitz, Duke University; Richard A. Gibbs, Fort Lewis College; M. G. Greening, University of New South Wales, Australia; Shiv Kumar and Miss Nirmal (jointly), Ohio State University; Problem Solving Group, Berne, Switzerland; Phil Tracy, Liverpool, New York; C. S. Venkataraman, Sree Kerala Varma College, Trichur, India; Kenneth Wilke, Topeka, Kansas; and the proposer.

Trigonometric Inequalities

835. [May, 1972] *Proposed by Sidney H. L. Kung, University of Jacksonville, Florida.*

Show that:

(a) $\sin(\cos x) < \cos(\sin x)$

(b) $\cos(\sin^{-1}x) < \sin^{-1}(\cos x)$, $0 \leqq x \leqq 1$

(c) $|\sin px| < p\,|\sin x|$ for any integral $p > 1$, $\sin x \neq 0$.

Solution by Leon Bankoff, Los Angeles, California.

(a) Let $\alpha = \cos x$. Then $\sin\alpha = \sin(\cos x)$. Since $\sin\alpha < \alpha$, we have $\sin(\cos x) < \cos x$. Similarly $\sin x < x$. So $\cos x < \cos(\sin x)$. It follows that $\sin(\cos x) < \cos x < \cos(\sin x)$.

(b) Let $\sin\alpha = \cos x$. Then $\alpha = \sin^{-1}(\cos x)$. But $\sin\alpha < \alpha$. Hence $\cos x < \sin^{-1}(\cos x)$. Similarly $x > \sin x$. So $\sin^{-1}x > \sin^{-1}(\sin x) \equiv x$. And $\cos(\sin^{-1}x) < \cos x$. Hence $\cos(\sin^{-1}x) < \cos x < \sin^{-1}(\cos x)$.

(c) Using induction and the identity

$$\sin 2kx/\sin x = 2[\cos x + \cos 3x + \cos 5x + \cdots + \cos(2k-1)x],$$

we obtain

$$\left|\frac{\sin 2kx}{\sin x}\right| < 2k$$

and

$$\left|\frac{\sin(2k+1)x}{\sin x}\right| < 2k+1.$$

The substitution $p = 2k$ yields

$$\left|\frac{\sin px}{\sin x}\right| < p.$$

Also solved by Edward D. Bender, Bradley University; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Barbara A. Keller, Fort Wayne, Indiana. Vaclav Konecny, Jarvis Christian College; Lew Kowarski, Morgan State College, Maryland; L. Lucetreiff, Northwestern University; Maurice Mizrahi, University of Texas at Austin; Edward Moylan, Plymouth, Michigan; Warren Page, New York City Community College; C. B. A. Peck, State College, Pennsylvania; Cecil G. Phipps, Tennessee Technological University; Qazi Zameeruddin, K. M. College, Delhi, India; and the proposer.

Euclidean Metric

836. [May, 1972] *Proposed by R. A. Struble, North Carolina State University at Raleigh.*

Find simple necessary and sufficient conditions on the function $f: R \to C$ from the real line to the complex plane in order that

$$d(a, b) = \sup\{|f(t + a) - f(t + b)| : t \in R\}$$

define a metric which induces the Euclidean topology for R.

Solution by the proposer.

Necessary and sufficient conditions are (i) f is uniformly continuous and either $(\text{ii})_a$ $\inf_{|\tau| \geqq \varepsilon} \sup\{|f(t+\tau) - f(t)| : t \in R\} > 0$ for every $\varepsilon > 0$ or $(\text{ii})_b$ f is nonperiodic and $\lim \inf_{|\tau| \to \infty} \sup\{|f(t+\tau) - f(t)| : t \in R\} > 0$. Indeed, the first of these (i) is necessary and sufficient for the induced topology to be coarser than the Euclidean topology and insures that $d(a, b) < \infty$. The second $(\text{ii})_a$ is necessary and sufficient for the induced topology to be finer than the Euclidean topology. The conditions in $(\text{ii})_b$ constitute an alternative to $(\text{ii})_a$ in conjunction with (i).

A Well-Known Triangle Property

837. [May, 1972] *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

Prove that the altitudes of any triangle bisect the angles of another triangle whose vertices are the feet of the altitudes of the first triangle.

Solution by Vaclav Konecny, Jarvis Christian College, Texas.

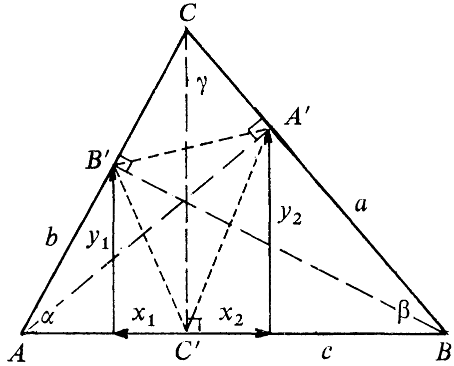
From the figure: $x_1 = b \cos \alpha - c \cos^2 \alpha$

$$= c\left(\frac{\sin \beta}{\sin \gamma} \cos \alpha - \cos^2 \alpha\right) = c \frac{\cos \alpha \sin \alpha}{\sin \gamma}\left(\frac{\sin \beta - \cos \alpha \sin(\alpha + \beta)}{\sin \alpha}\right)$$

$$= -c \frac{\cos \alpha \sin \alpha}{\sin \gamma} \cos(\alpha + \beta).$$

$$y_2 = c \cos \beta \sin \beta.$$

x_2 and y_1 are obtained from x_1 and y_2 resp. by putting α for β and β for α. Put $x_1 y_2 = f(\alpha, \beta) \cdot f(\alpha, \beta) = f(\beta, \alpha)$ thus $x_1 y_2 = x_2 y_1$. So the height dropped from C on c bisects $\measuredangle B'C'A'$. The same procedure applies for A' and B'. And the proof follows for any type of a plane triangle provided that $\alpha, \beta, \gamma \neq 0$.



Also solved by Leon Bankoff, Los Angeles, California; C. R. J. Clapham, University of Aberdeen, Scotland; W. M. Clements, Grand Junction, Colorado; Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, New York; Clayton W. Dodge, University of Maine; Ragnar Dybvik, Tingvoll, Norway; Abraham L. Epstein, Bedford, Massachusetts; Gabriel V. Ferrer, Universidad Autonoma de Baja, California; George Fabian, Park Forest, Illinois; Herta T. Freitag, Hollins, Viriginia; Michael Goldberg, Washington, D. C.; Lester Rubenfeld for Michael Frame, David Hale, Clayton Keller, David Rutledge, Stephen Sentoff and David Wagner (Jointly), Rensselaer Polytechnic Institute; Roger Izard, Dallas, Texas; Barbara A. Keller, Fort Wayne, Indiana; J.D.E. Konhauser, Macalester College, St. Paul, Minnesota; Lew Kowarski, Morgan State College. Maryland; Shiv Kumar and Miss Nirmal (jointly), Ohio State University; Maurice Mizrahi, University of Texas at Austin; J. Paul Moulton, Temple University; C. C. Oursler, Belleville, Illinois; Lawrence A. Ringenberg, Eastern Illinois University; Gerald V. Rowell, Moorhead, Minnesota; Phil Tracy, Liverpool, New York; Charles W. Trigg, San Diego, California; Zalman Usiskin, University of Chicago; Dimitrios Vathis, Chalcis, Greece; C. S. Venkataraman, Sree Kerala Varma College, Trichur, India; William Wernick, City College, New York; Qazi Zameeruddin, K. M. College, Delhi, India; and the proposer.

Comment on Q484

Q484. [September, 1970] Show that the length of one leg of a Pythagorean triangle must be a multiple of 3.

[*Submitted by Charles W. Trigg*]

Comment by Eric C. Nummela, University of Florida.

For any integer x, x^2 is congruent to 0 or 1 modulo 3. Thus the only solutions in integers modulo 3 of the equation $a^2 + b^2 = c^2$ are

$$0 + 0 = 0,\ 1 + 0 = 1,\ 0 + 1 = 1.$$

In any case at least one of a or b must be congruent to 0 modulo 3.

Comment on Q543

Q543. [May, 1972] Show that for all natural numbers $n \geqq 4$, $(n-1)^n > n^{n-1}$.

[*Submitted by Alexander Zujus*]

I. *Comment by K. R. S. Sastry, Makele, Ethiopia.*

For $n > 1$, $n - 1 > 0$

$$\therefore\ (n-1)^n > n^{n-1} \Leftrightarrow n - 1 > \left(\frac{n}{n-1}\right)^{n-1}.$$

But

$$\left(\frac{n}{n-1}\right)^{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} \to e; \qquad n \to \infty$$

and

$$\left(\frac{n}{n-1}\right)^{n-1} \text{ is bounded.}$$

$$\therefore\ (n-1)^n > n^{n-1} \text{ is true}$$

for all *real numbers* $> e + 1$.

II. *Comment by Murray S. Klamkin, Ford Motor Company.*

More generally, $x^{1/x}$ is a monotonic decreasing function for $x \geqq e$. This follows since

$$Dx^{1/x} = x^{1/x}(1 - \log x)/x^2.$$

ANSWERS

A562. Suppose $f(z)$ is not constant and let $M_n = \max_{|z|=n} |f(z)|$. Since the Maximal Principle required that $\sup_{|z| \leqq n} |f(z)|$ occurs strictly on the boundary, we have

$$M \underset{n \to \infty}{\to} \infty.$$

A563. Since p has primitive roots, it follows that G is cyclic and therefore H is cyclic. If the order of H is r, then the elements of H consist of $\{1, a, a^2, \cdots a^{r-1}\}$ for some a in G.
Then

$$\sum_{k=0}^{r-1} a^k = \frac{a^r - 1}{a - 1} \equiv 0 \pmod p \text{ since } a^r \equiv 1 \pmod p.$$

Therefore

$$p \mid \sum_{k=0}^{r-1} a^k.$$

A564. Let $A_iB_i = R_{\sim 1}$, $B_iC_i = S_{\sim 1}$, $C_iD_i = T_{\sim 1}$, $OA_i = U_{\sim 1}$

$(i = 1, 2, 3, 4)$ then the given vectors can be shown to reduce to

$$R_{\sim i} + T_{\sim 1} + R_{\sim i+1} + T_{\sim i+1} + R_{\sim i+2} + T_{\sim i+2} \qquad (i = 1, 2, 3, 4).$$

Consequently $R_{\sim i} + T_{\sim i} = 0$ which is invariant under rigid body motions. Also, the quadrilaterals must be parallelograms.

A565. It follows by continuity that none of the face angles can be acute or obtuse. Thus the only possibility is a trirectangular angle. If $OA = a$, $OB = b$, $OC = c$, then

$$AB^2 = a^2 + b^2, \; BC^2 = b^2 + c^2, \; AC^2 = c^2 + a^2.$$

Since the sum of any two is greater than the third, ABC is acute.

A566. No. Twin primes greater than three have the forms $6k - 1$ and $6k + 1$, so their mean is the even integer $6k$. Even perfect numbers have the form $2^{n-1}(2^n - 1)$ with $2^n - 1$ a prime. If $6k$ is perfect, k is a power of 2 and $2^n - 1 = 3$. Thus $n = 2$ and 6 is the unique perfect mean of twin primes.

(Quickies on page 104)

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TOPICS FROM TRIANGLE GEOMETRY

By D. Moody Bailey

Consider the following ratios when P is a point in the plane of triangle ABC, with DEF its cevian triangle.

(a)
$$\frac{BD}{DC}=\left(\frac{a-b}{c-a}\right)^2\left(\frac{a+b-c}{a+c-b}\right),\quad \frac{CE}{EA}=\left(\frac{b-c}{a-b}\right)^2\left(\frac{b+c-a}{a+b-c}\right),$$
$$\frac{AF}{FB}=\left(\frac{c-a}{b-c}\right)^2\left(\frac{a+c-b}{b+c-a}\right).$$

(b)
$$\frac{BD}{DC}=\frac{c^2}{b^2}\left(\frac{a^2+b^2-2c^2}{a^2+c^2-2b^2}\right),\quad \frac{CE}{EA}=\frac{a^2}{c^2}\left(\frac{b^2+c^2-2a^2}{a^2+b^2-2c^2}\right),\quad \frac{AF}{FB}=\frac{b^2}{a^2}\left(\frac{a^2+c^2-2b^2}{b^2+c^2-2a^2}\right).$$

(c)
$$\frac{BD}{DC}=\frac{a^2+c^2-b^2-ac}{a^2+b^2-c^2-ab},\quad \frac{CE}{EA}=\frac{a^2+b^2-c^2-ab}{b^2+c^2-a^2-bc},\quad \frac{AF}{FB}=\frac{b^2+c^2-a^2-bc}{a^2+c^2-b^2-ac}.$$

(d)
$$\frac{BM}{MC}=-\left[\frac{a^4+c^4(CE/EA)-2a^2c^2(CD/DB)}{b^4+a^4(AF/FB)-2a^2b^2(AE/EC)}\right]\frac{AE}{EC},$$
$$\frac{CN}{NA}=-\left[\frac{b^4+a^4(AF/FB)-2a^2b^2(AE/EC)}{c^4+b^4(BD/DC)-2b^2c^2(BF/FA)}\right]\frac{BF}{FA},$$
$$\frac{AO}{OB}=-\left[\frac{c^4+b^4(BD/DC)-2b^2c^2(BF/FA)}{a^4+c^4(CE/EA)-2a^2c^2(CD/DB)}\right]\frac{CD}{DB}.$$

Can you demonstrate that (a) fixes the famous Feuerbach point, (b) determines the inverse of the symmedian point with respect to the circumcircle, (c) gives ratio values for the reflective partner of the incenter, and (d) fixes line MNO that is the polar of point P with respect to the Brocard circle? Is it true that the incenter and Gergonne point are the centers of similitude for the two internal Soddy circles? What about the centers of similitude for the three pairs of external Soddy circles? Could you write ratio values for line MNO when it becomes the radical axis of the Brocard and nine-point circles? Is it possible to write cevian ratios for a point on the Euler line that divides the segment connecting the orthocenter and de Longchamps' point in a given ratio? The preceding examples may be solved by considering special cases of general results derived by the author.

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